Lecture 1

Structure of stars:

- Hydrostatic equilibrium.
- Energy transport.
- Energy generation.

1) Mass conservation:

Let $M(r)$ be mass contained in a given radius 'r'. Then the mass contained between r and $r+dr$ is:

$$M(r+dr) - M(r) = \rho(r)\, dV$$

with $dV = 4\pi r^2 dr$, and $\rho(r)$ the density. Then

$$\frac{dM}{dr} dr = \rho\, 4\pi r^2 dr \Rightarrow \boxed{\frac{dM_r}{dr} = \rho(r)\, 4\pi r^2}$$

where $M_r = M(r)$.

2) Hydrostatic equilibrium

Balance between gravity and pressure of the fluid.

- Pressure: Let us consider a shell of area dA. The force is given by (force positive if pointing outwards):

$$F_p = P(r)\, dA - P(r+dr)\, dA = \dots = -\frac{dP}{dr} dr\, dA$$

- Gravity:

$$F_G = -\frac{GM_r\, dm}{r^2}$$

with $dm = \rho(r)\, dV = \rho(r)\, dA\, dr$

$$\left.\right\} F_G = -\frac{GM_r\, \rho(r) dA\, dr}{r^2}$$

- Newton's law:

$$\underbrace{m(r)}_{\rho(r) dA dr} \frac{d^2 r}{dt^2} = -\frac{dP}{dr} dr\, dA - \frac{GM_r\, \rho(r)\, dA\, dr}{r^2}$$

$$\rho(r) \frac{d^2 r}{dt^2} = -\frac{dP}{dr} - \frac{GM_r\, \rho(r)}{r^2}$$

In hydrostatic equilibrium: $\frac{d^2 r}{dt^2} = 0$

$$\boxed{\frac{dP}{dr} = -\frac{GM_r\, \rho(r)}{r^2}}$$

3) Energy conservation

Energy loss via radiation. Generation of energy is needed (nuclear reactions...)

$L(r)$: energy flow across a spher of radius r (Energy/time)

$$L(r+dr) - L(r) = \frac{dL(r)}{dr}\, dr$$

ϵ : energy generation rate per mass unit (Energy/(time.mass))

$$\epsilon\, dm = \epsilon \rho(r)\, 4\pi r^2\, dr$$

Thermal equilibrium : radiation loss must equal the energy gain :

$$\frac{dL_r}{dr} = \epsilon \rho(r)\, 4\pi r^2$$

we denote: $L_r \equiv L(r)$

ϵ depends on the density, temperature and chemical composition

$$\epsilon = \epsilon(\rho, T, X_i)$$

4) Energy transport

Three possibilities: conduction, radiation and convection

In conduction and radiation, the $\overbrace{\text{flux of energy}}^{f(r)}$ flow (Energy/(Area·time)) is proportional to the temperature gradient:

$$f(r) = -\lambda \frac{dT(r)}{dr}$$

with λ the conductivity coefficient.

- $\lambda > 0$ :: transfer from high temperature regions to low temp.-ones.
- $\lambda = \lambda(\rho, T, X_i)$

It holds: $L_r = 4\pi r^2 f(r)$

so that :

$$\boxed{\frac{dT}{dr} = -\frac{1}{4\pi r^2 \lambda} L_r}$$

We leave apart here the bad understood convection mechanism.

Summary:

$$\boxed{\begin{aligned} \frac{dM_r}{dr} &= 4\pi r^2 \rho \\ \frac{dP}{dr} &= -\frac{GM_r}{r^2}\rho \\ \frac{dL_r}{dr} &= 4\pi r^2 \epsilon \rho \\ \frac{dT}{dr} &= -\frac{1}{4\pi r^2 \lambda} L_r \end{aligned}}$$

Primary variables: M_r, P_r, L_r, T as functions of r.

Need of auxiliary equations:

- ρ: equation of state $P = P(\rho, T, X_i)$
- λ: coefficient of conductivity. $\lambda = \lambda(\rho, T, X_i)$
- ϵ: nuclear fusion rate, $\epsilon(\rho, T, X_i)$

5) The radius is a bad parameter to set boundary conditions, (the total radius is a priori not known). It is better to use the mass, since the total mass is a good parameter of the resulting star. We can write

$$\boxed{\begin{aligned} \frac{dr}{dM_r} &= \frac{1}{4\pi r^2 \rho} \\ \frac{dP}{dM_r} &= -\frac{GM_r}{4\pi r^4} \\ \frac{dL_r}{dM_r} &= \epsilon \\ \frac{dT}{dM_r} &= -\frac{1}{16\pi^2 r^4 \lambda \rho} L_r \end{aligned}}$$

($\rho > 0$, then one-to-one relation $r - M_r$)

Invert eq. of state: $\rho = \rho(P, T, X_i)$
so that r.h.s. are given in terms of M_r, P, T, L_r (and X_i) parameter

Boundary conditions at the center: $M_r = 0 \longrightarrow r = 0, L_r = 0$

Boundary conditions at the surface: $M_r = M \longrightarrow \underbrace{P = 0,\ T = 0}_{\text{naive and not very good.}}$ $\Big\} \to$ need of atmosphere

- Energy conservation ~~implies~~ suggests gravitational redshift (gedanken experiment)

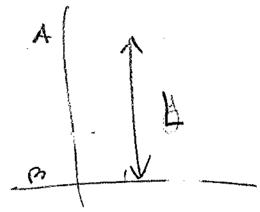
Let us consider a particle of mass m in a constant gravitational field at a height L

Let us consider it in a Newtonian treatment.

1) Initially at rest, at A its rest energy is:

$$E = mc^2$$

2) It falls to B so that its energy there:



$$E = mc^2 + \underbrace{mgL}_{\text{kin. energ.}}$$

3) At B it gets annihilated producing a photon that goes back upwards to A with energy:

$$E_{photon} = mc^2 + mgL$$

If $E_{top} = mc^2 + mgL$ then we can create a particle of mass m plus an excess of energy mgh.

(the key thing is that the whole photon energy can be converted into rest mass + extra kinetic, thermal ... energy: it is not a potential energy but it is "located" at the photon, therefore:

$$E_{p.tot\,A} = mc^2 + \underbrace{mgL}_{pot} \qquad E_{phot\,A} = mc^2 + mgL \longrightarrow E_{p.tot\,A} = mc^2 + \underbrace{mgL}_{thermal}$$

$$\downarrow \qquad\qquad\qquad \uparrow$$

$$E_{p.tot\,B} = mc^2 + \underbrace{mgL}_{kin} \longrightarrow E_{phot\,B} = mc^2 + mgL$$

The key difference between mgh_{pot} y $mgh_{thermal}$ at A is that the latter is localized an be "used", whereas the former is referencial (non-local) and cannot be used (it is potential).

I would prefer to argue that $E_{phot\,B} \to E_{phot\,A}$ is not possible because it produces a machine of creation of energy

$$E_{rest\,A} = mc^2 \qquad E_{phot\,A} = mc^2 + mgL \longrightarrow E_{rest\,A}\ \overbrace{m\left(1 + \frac{gL}{c^2}\right)}^{m_2} c^2$$

$$\downarrow \qquad\qquad \uparrow \qquad\qquad \downarrow$$

$$E_{rest+kin\,B} = mc^2 + mgL \longrightarrow E_{phot\,B} = mc^2 + mgL \qquad E_{rest+kin\,B} = mc^2 + 2mgL \longrightarrow E_{phot\,B}$$

n times

$$\longrightarrow \quad E_{rest+kin} = mc^2 + n\,mgL$$

The way out is to accept that the photon looses energy when going from B to A:

$E_{phot\,B} = mc^2 + mgy \longrightarrow E_{phot\,A} = mc^2$

$$E_{phot\,B} = mc^2\left(1 + \frac{gy}{c^2}\right)$$

Using $E = h\nu (= \hbar\omega)$, $\lambda = \frac{c}{\nu}$ $z = \frac{\lambda_A - \lambda_B}{\lambda_B}$; $1+z = \frac{\lambda_A}{\lambda_B}$

$$\boxed{1+z = \frac{\lambda_A}{\lambda_B} = \frac{\nu_B}{\nu_A} = \frac{h\nu_B}{h\nu_A} = \frac{E_B}{E_A} = 1 + \frac{gy}{c^2}}$$

$$\boxed{z = \frac{gy}{c^2}} \qquad \text{(Einstein 1911)}$$

Verified by Pound & Rebka (1960)

- Principle of equivalence implies gravitational red shift.

Principle of equivalence (Einstein 1908, 1911): "all effects of a uniform gravitational field are identical to the effects of a uniform acceleration of the coordinate system".

generalised

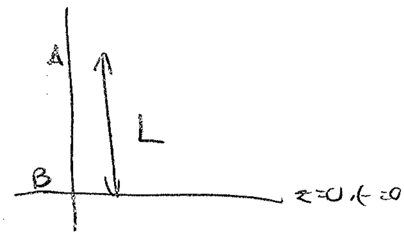
Using this, let us deduce that a gravitational field must produce redshift.

Consider again A and B, and A emits a photon.
According to the equivalence principle we can consider them accelerated at $a = g$ (in a rocket without grav. field; notes the sign of the acceleration is opposed)

$$\boxed{m\frac{d^2x}{dt^2} = F = mg \; ; \quad \frac{d^2x}{dt^2} - g = 0; \quad \frac{d^2x'}{dt^2} = 0}$$

$$\boxed{x' = x - \tfrac{1}{2}gt^2 = x + \tfrac{1}{2}at^2 \quad \boxed{a = -g}}$$

We neglect 2nd order terms $\left(\frac{v}{c}\right)^2$ $\left(\frac{gy}{c^2}\right)^2$



Position A: $z_A = L + \frac{1}{2}gt^2$

Position B: $z_B = \frac{1}{2}gt^2$

1) A sents a first photon at $t=0$, that B receives at $t=t_1$

the distance is:

$$\left.\begin{array}{l}\underbrace{z_A(0)-z_B(t_1)}_{L+0-\frac{1}{2}gt_1^2}=ct_1\end{array}\right\}\quad \boxed{L-\frac{1}{2}gt_1^2=ct_1} \quad (*)$$

2) A 2nd photon at $t=\Delta\tau_A$ and B receives at $\Delta\tau_B$ after receiving the first, i.e. $t_2=t_1+\Delta\tau_B$

Then the distance travelled by the 2nd photon is:

$$z_A(\Delta\tau_A)-z_B(t_1+\Delta\tau_B) = c(t_2-\Delta\tau_A)=c(t_1+\Delta\tau_B-\Delta\tau_A)$$

$$= L+\frac{1}{2}g(\Delta\tau_A)^2-\frac{1}{2}g(t_1+\Delta\tau_B)^2$$

$$= L+\frac{1}{2}g\cancel{\Delta\tau_A^2}^{\,0}-\underline{\frac{1}{2}gt_1^2}-g\underline{t_1\Delta\tau_B}-\frac{1}{2}g\cancel{\Delta\tau_B^2}^{\,0}$$

$$\simeq L-\frac{1}{2}gt_1^2-gt_1\Delta\tau_B$$

$$\boxed{L-\frac{1}{2}gt_1^2-gt_1\Delta\tau_B=c(t_1+\Delta\tau_B-\Delta\tau_A)} \quad (**)$$

Subtracting $(**)-(*)$:

$$-gt_1\Delta\tau_B=c(\Delta\tau_B-\Delta\tau_A)$$

Using from $(*)$ $t_1\simeq\frac{L}{c}$ we have

$$\Delta\tau_A=\Delta\tau_B\left(1+\frac{gt_1}{c}\right)=\Delta\tau_B\left(1+\frac{gL}{c^2}\right)$$

In terms of frequencies: $\nu\sim\frac{1}{\Delta\tau}$

so we have:

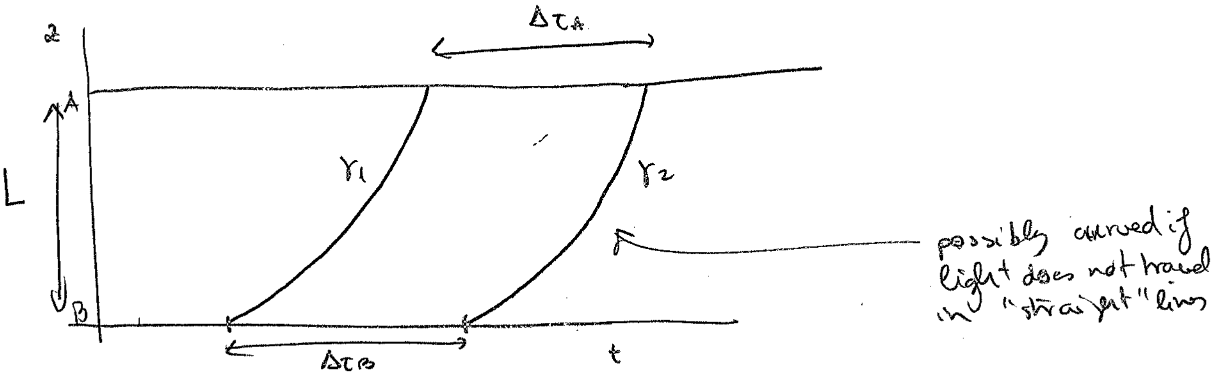
$$\nu_B=\nu_A\left(1+\frac{gL}{c^2}\right)$$

$$1+z=\frac{\lambda_A}{\lambda_B}=\frac{\nu_B}{\nu_A}=1+\frac{gL}{c^2} \qquad \boxed{z=\frac{gL}{c^2}}$$

as in page 5.

- Gravitational redshift is inconsistent with flat spacetime in special relativity

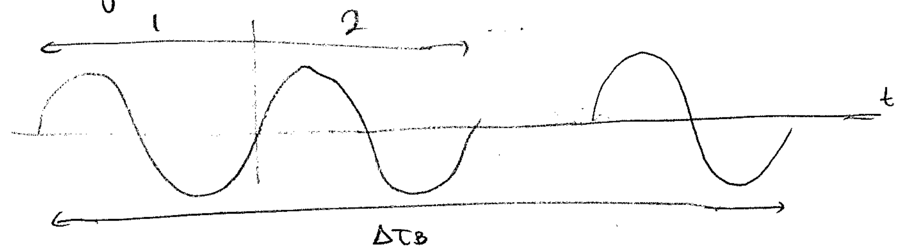
Schild (1960, 1962, 1967)



Argument in special relativity: independent of the mathematical nature of gravitational field.

Observers A and B at rest with respect to each other and with respect the Earth: use for instance radar ranging to rest particles far from the Earth

Emitter sends signal at frequency ν_B. Let assume that it is a periodic signal with N cycles. Then $N = \nu_B \Delta\tau_B$ $\quad (\nu_B = \frac{1}{T_B})$



The receiver at A receives the N cycles in a time $\Delta\tau_A$ so that: $N = \nu_A \Delta\tau_A$

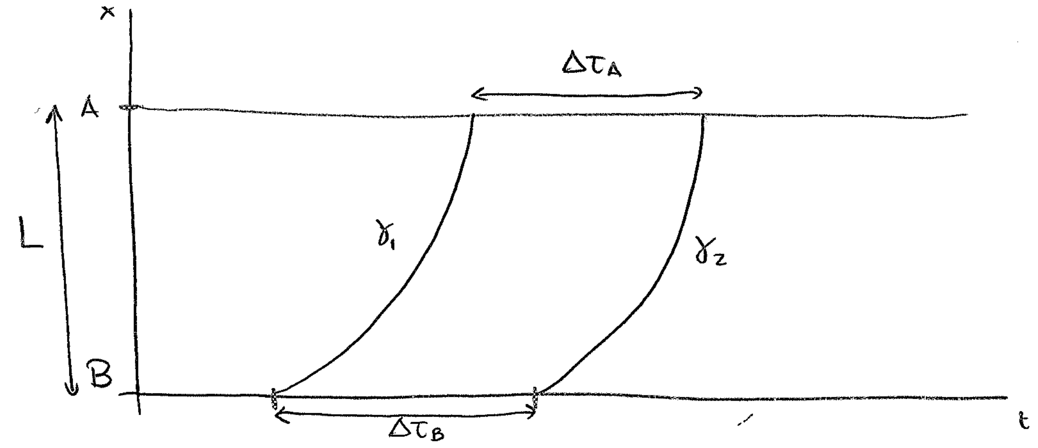
. Therefore $\nu_A \Delta\tau_A = \nu_B \Delta\tau_B$

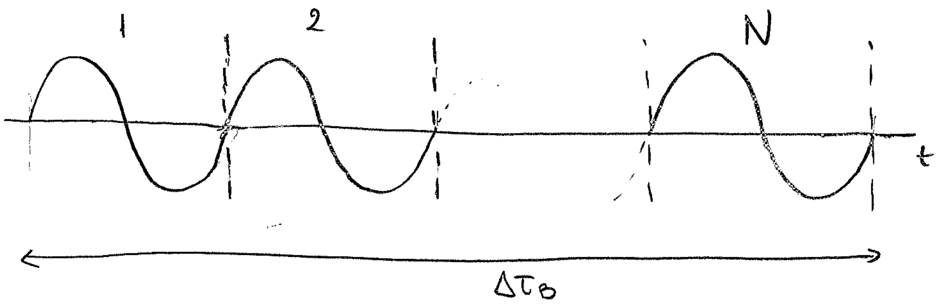
If the redshift is present:

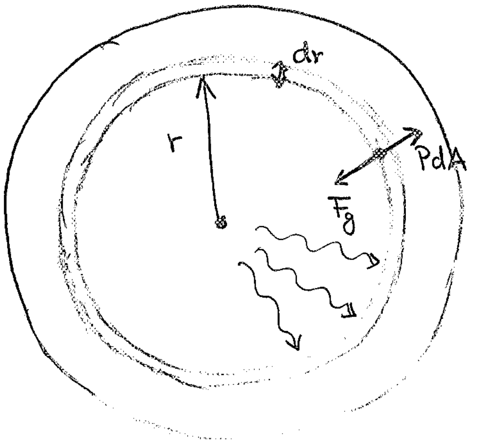
$$\nu_A > \nu_B$$

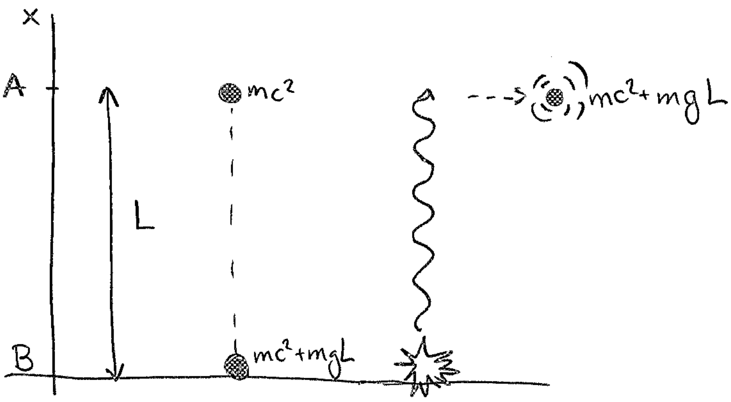
so that $\boxed{\Delta\tau_B < \Delta\tau_A}$

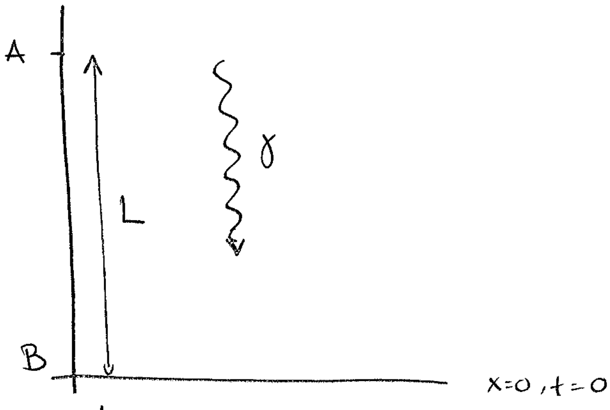
But since the gravitational field is static, the observers do not move, curves γ_1 and γ_2 must be "congruent" (the "same" except for the position in space). Therefore, in a "flat" space and time diagram (spacetime), they must form a "parallelogram", so that $\Delta\tau_A = \Delta\tau_B$.
This contradiction indicates that flat special relativity (Minkowski) spacetime is not adequate for the description of gravity (if the gravitational redshift is present)

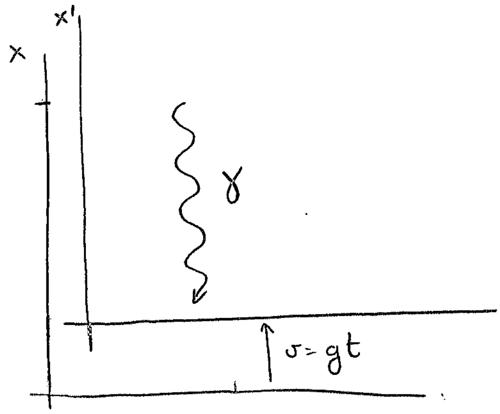

x
A
B
L
$\Delta\tau_A$
γ_1
γ_2
$\Delta\tau_B$
t

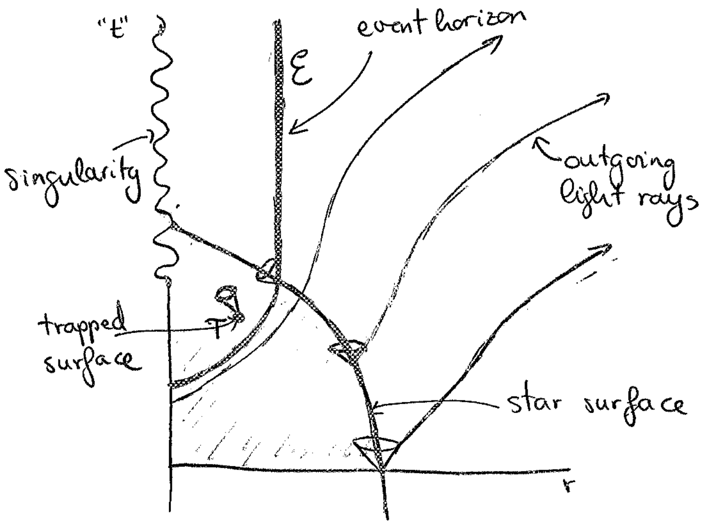

1
2
N
t
$\Delta\tau_B$

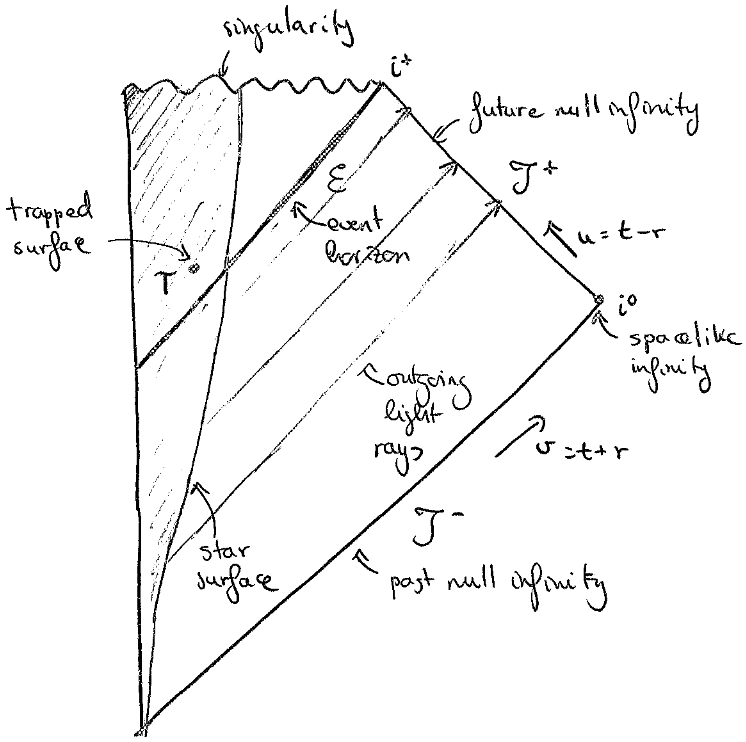

dr
r
PdA
F_g


x
A
B
L
mc^2
mc^2+mgL
mc^2+mgL


A
B
L
γ
$x=0, t=0$


x
x'
γ
$v=gt$


"t"
event horizon
$\mathcal{E}$
singularity
outgoing light rays
trapped surface
T
star surface
r


singularity
i^+
future null infinity
$\mathcal{J}^+$
$\mathcal{E}$
event horizon
trapped surface
T
$u = t - r$
i^0
spacelike infinity
outgoing light rays
$v = t + r$
$\mathcal{J}^-$
star surface
past null infinity

1.2. Lie derivative

1.2.1. Passive aspects on tensor transformations : coordinate change

Let T be a $\binom{m}{n}$ (m-times contravariant, n-times covariant tensor)

Let $\{x^\mu\}$ be a coordinates system. Then:

$$T = T^{\mu_1 \cdots \mu_m}{}_{\nu_1 \cdots \nu_n} \frac{\partial}{\partial x^{\mu_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{\mu_m}} \otimes dx^{\nu_1} \otimes \cdots \otimes dx^{\nu_n}$$

Let us consider the invertible change of coordinates to coordinates $\{y^\mu\}$:

$$y^\mu = y^\mu(\{x^\nu\}) \; ; \quad \left(\begin{array}{ll} y^0 = y^0(x^0, x^1, x^2, x^3) & x^0 = x^0(y^0, y^1, y^2, y^3) \\ y^1 = y^1(x^0, x^1, x^2, x^3) \quad \text{inverse:} & y^1 = y^1(x^0, x^1, x^2, x^3) \\ y^2 = y^2(x^0, x^1, x^2, x^3) & y^2 = y^2(x^0, x^1, x^2, x^3) \\ y^3 = y^3(x^0, x^1, x^2, x^3) \; ; & y^3 = y^3(x^0, x^1, x^2, x^3) \end{array} \right)$$

$$x^\mu = x^\mu(\{y^\nu\})$$

Then:

$$T = T^{\mu_1 \cdots \mu_m}{}_{\nu_1 \cdots \nu_n} \left(\frac{\partial y^{\rho_1}}{\partial x^{\mu_1}}\right) \cdots \left(\frac{\partial y^{\rho_m}}{\partial x^{\mu_m}}\right) \left(\frac{\partial x^{\nu_1}}{\partial y^{\sigma_1}}\right) \cdots \left(\frac{\partial x^{\nu_n}}{\partial y^{\sigma_n}}\right) \frac{\partial}{\partial y^{\rho_1}} \otimes \cdots \otimes \frac{\partial}{\partial y^{\rho_m}} \otimes dy^{\sigma_1} \otimes \cdots \otimes dy^{\sigma_n}$$

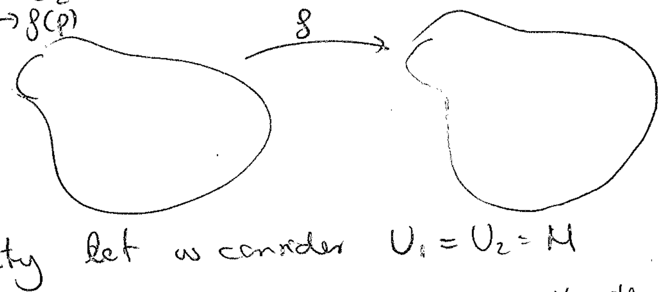
Therefore $\{x^\mu\} \to \{x'^\mu\}$

$$\boxed{T'^{\mu_1 \cdots \mu_m}{}_{\nu_1 \cdots \nu_n} = \left(\frac{\partial x'^{\mu_1}}{\partial x^{\rho_1}}\right) \cdots \left(\frac{\partial x'^{\mu_m}}{\partial x^{\rho_m}}\right) \left(\frac{\partial x^{\sigma_1}}{\partial x'^{\nu_1}}\right) \cdots \left(\frac{\partial x^{\sigma_n}}{\partial x'^{\nu_n}}\right) T^{\rho_1 \cdots \rho_m}{}_{\sigma_1 \cdots \sigma_n}}$$

1.2.2 Active interpretation.

Let us consider an invertible differentiable mapping (diffeomorphism).

$$f : U_1 \subset M \to U_2 \subset M$$
$$\quad p \mapsto f(p)$$



For simplicity let us consider $U_1 = U_2 = M$

In a given coordinate chart, we can write the mapping as:

$$x'^\mu = f^\mu(x^\nu) \qquad \left(\begin{array}{l} x'^0 = f^0(x^0, x^1, x^2, x^3) \\ \vdots \\ x'^3 = f^3(x^0, \ldots, x^3) \end{array} \right)$$

$$(= x'^\mu(x^\nu))$$

We can also write the inverse:

$x^\mu = f^{-1\mu}(x'^\nu)$

Push-forward of a scalar ⟶ can $M_1 \neq M_2$

The differential of f, Df, and of f^{-1}, $Df^{-1} = (Df)^{-1}$, are linear applications

$$Df : TM \to TM'$$
$$v \mapsto Df(v) = v'$$

such that

$$v'^\mu = \left(\frac{Df^\mu}{\partial x^\nu}\right) v^\nu$$

in the coordinate system $\{x^\mu\}$.

• Pull-back

We define the pull-back of a $\binom{0}{1}$ tensor α, $f^*(\alpha)$:

$$\forall v,\quad f^*\alpha(v) = \alpha(Df\, v) = \alpha_\nu \left(\frac{\partial f^\nu}{\partial x^\mu}\right) v^\mu ; \quad \boxed{(f^*\alpha)_\mu = \left(\frac{\partial f^\nu}{\partial x^\mu}\right)\alpha_\nu}$$

$$\left(\frac{\partial x'^\nu}{\partial x^\mu}\right)\alpha_\nu$$

Push-forward of $\binom{1}{0}$ v, $f_* v$:

$$\forall \alpha,\quad f_* v(\alpha) = v(f^*\alpha) = v^\mu \left(\frac{\partial f^\nu}{\partial x^\mu}\right)\alpha_\nu ; \quad \boxed{(f_* v)^\nu = \left(\frac{\partial f^\nu}{\partial x^\mu}\right) v^\mu} ; \quad (f_* v) = Df(v)$$

$$\left(\frac{\partial x'^\nu}{\partial x^\mu}\right) v^\mu$$

When having an inverse f^{-1}, then: $\left(\frac{\partial f^\nu}{\partial x^\mu}\right)^{-1}$

$$f_*\alpha = (f^{-1})^*\alpha \Rightarrow ((f^{-1})^*\alpha)_\mu = \left(\frac{\partial f^{-1\nu}}{\partial x'^\mu}\right)\alpha_\nu = \left(\frac{\partial x'^\nu}{\partial x^\mu}\right)^{-1}\alpha_\nu = \left(\frac{\partial x^\nu}{\partial x'^\mu}\right)\alpha_\nu$$

$$f^* v = (f^{-1})_* v \to ((f^{-1})_* v)^\mu = \left(\frac{\partial f^{-1\mu}}{\partial x^\nu}\right) v^\nu = \left(\frac{\partial x'^\mu}{\partial x^\nu}\right)^{-1} v^\nu = \left(\frac{\partial x^\mu}{\partial x'^\nu}\right) v^\nu$$

For a scalar $\varphi : M \to \mathbb{R}$

$$f_*\varphi(x) \equiv \varphi(f^{-1}(x')) \Rightarrow \boxed{f_*\varphi(f(x)) = \varphi(x) ; \; f_*\varphi(x') = \varphi(x)}$$

In general, for a Tensor T, the pushforward $f_* T$:

$$\boxed{(f_* T)^{\mu_1 \dots \mu_m}{}_{\nu_1 \dots \nu_n}(x') = \left(\frac{\partial x'^{\mu_1}}{\partial x^{\rho_1}}\right) \cdots \left(\frac{\partial x'^{\mu_m}}{\partial x^{\rho_m}}\right)\left(\frac{\partial x^{\sigma_1}}{\partial x'^{\nu_1}}\right) \cdots \left(\frac{\partial x^{\sigma_n}}{\partial x'^{\nu_n}}\right) T^{\rho_1 \dots \rho_m}{}_{\sigma_1 \dots \sigma_n}(x)}$$

Active / passive:

Passive		Active
(x'^μ) point p in $\{x'^\mu\}$	$x'^0 = x'^0(x^0, \dots x^3)$	(x^μ) coordinates of p in $\{x^\mu\}$
(x^μ) " " " $\{x^\mu\}$	$\vdots$	(x'^μ) " " $f(p)$ " $\{x^\mu\}$
	$x'^3 = x'^3(x^0, \dots x^3)$	

The tensor $f_* T$ at points $f(p)$ IS the tensor T as "it was" in $f^{-1}(p)$, and then transformed by f.

For instance, to calculate the pushforward of V of type $\binom{1}{1}$ at $f(x)$,

1) Write V at x:

$$V = V^\mu{}_\nu(x) \frac{\partial}{\partial x^\mu} \otimes dx^\nu$$

2) Rewrite in terms of $x'^\mu = x'^\mu(x^\nu)$

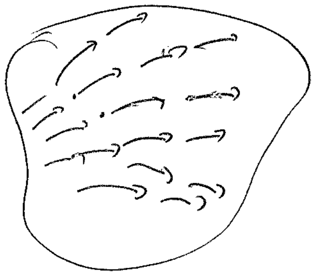
$$V = V^\mu{}_\nu(x(x')) \left(\frac{\partial x'^\rho}{\partial x^\mu}\right)\left(\frac{\partial x^\nu}{\partial x'^\sigma}\right) \frac{\partial}{\partial x'^\rho} \otimes dx'^\sigma$$

3) Identify with:

$$V'(x) = V'^\mu{}_\nu(x') \frac{\partial}{\partial x'^\mu} \otimes dx'^\nu$$

Then:
$$\boxed{V'^\mu{}_\nu(x') = \left(\frac{\partial x'^\mu}{\partial x^\rho}\right)(x) \left(\frac{\partial x^\sigma}{\partial x'^\nu}\right)(x) V^\rho{}_\sigma(x)}$$

1.2.3. Infinitesimal transformations.



Consider a vector $X = X^\mu \frac{\partial}{\partial x^\mu}$

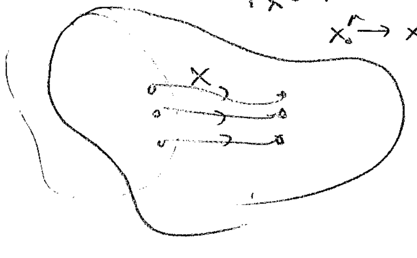
We define the flow of X by solving:

$$\frac{dx^\mu}{d\lambda} = X^\mu$$

that results in $x^\mu = x^\mu(\lambda; x_0^\mu)$, with x_0^μ the initial value.

For a given λ, varying x_0^μ we obtain a mappin

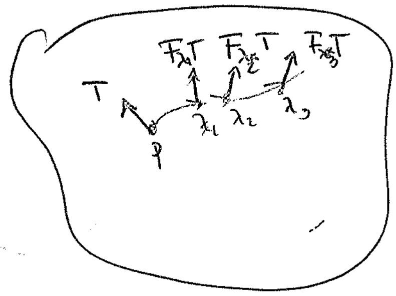
$$F_\lambda : M \to M$$
$$x_0^\mu \mapsto x^\mu(\lambda; x_0)$$



F_λ are diffeos of M, and X^μ is the infinitesimal generator.

We can consider the pull-back (or pushforward) of a tensor T along F_λ:

$F_\lambda^* T$



Lie-derivative:

In order to calculate a derivative of a linear object, we need to be able to substract its values in two close points. We cannot take the difference of T at p and T at p', because the live in different spaces: T_pM, $T_{p'}M$.

We need a rule to "transport" objects to the same point, where we can differentiate.

Thee infinitesimal difeo X^μ, provides such a rule, by bringing T at $f(p)$ to p (as it was in $f(p)$)

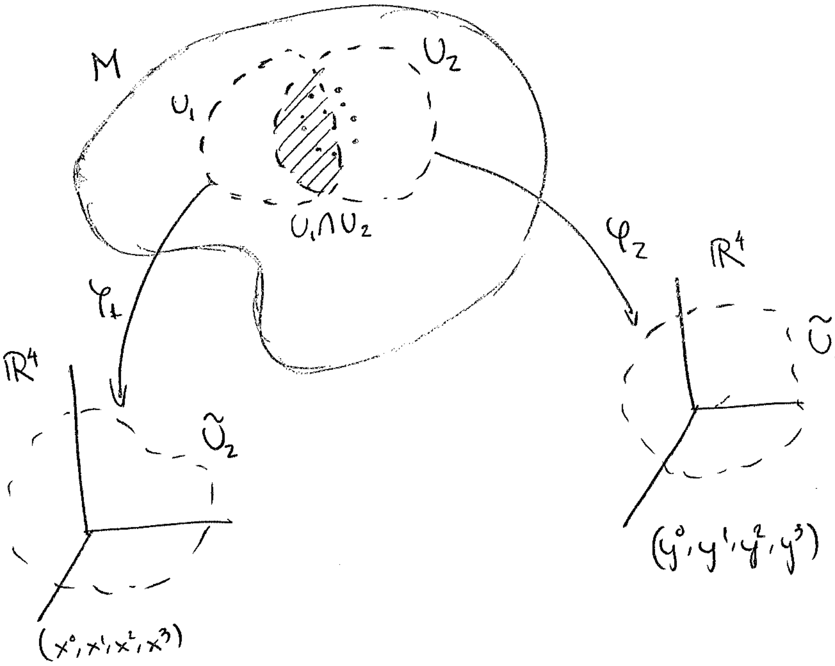
$$\boxed{\mathcal{L}_X T := \lim_{\lambda\to 0} \frac{F_\lambda^* T - T}{\lambda}}$$

A particularly useful expression is that, for any (torsion-free) derivative operator $\tilde{\nabla}$ we can write (eg. Wald 84. (C.2.14))

$$\boxed{\mathcal{L}_X T^{\mu_1\dots\mu_m}{}_{\nu_1\dots\nu_n} = X^\rho \tilde{\nabla}_\rho T^{\mu_1\dots\mu_m}{}_{\nu_1\dots\nu_n} - \sum_{k=1}^{m} T^{\mu_1\dots\rho\dots\mu_m}{}_{\nu_1\dots\nu_n} \tilde{\nabla}_\rho X^{\mu_k} + \sum_{k=1}^{n} T^{\mu_1\dots\mu_m}{}_{\nu_1\dots\rho\dots\nu_n} \tilde{\nabla}_{\nu_k} X^\rho}$$

$\tilde{\nabla}$ can be the partial derivative ∂_μ in a coordinate system or the covariant derivative that we discuss now.

Note that we only need i) a differentiable structure to define the Lie derivative, and ii) A vector X along which we derive.

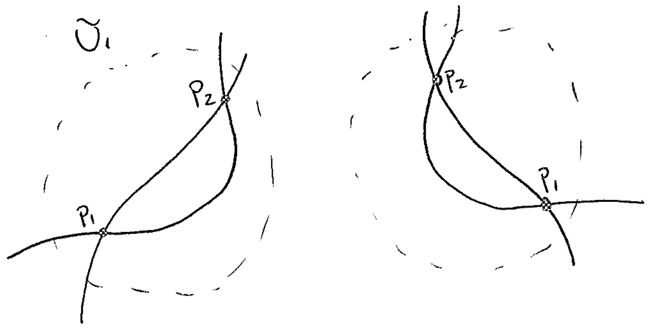


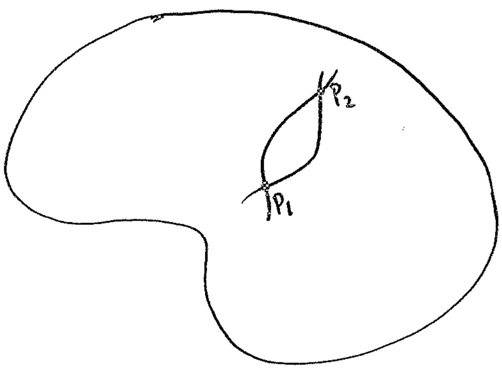
$$\varphi_1 : U_1 \to \tilde{U}_1 \subset \mathbb{R}^4$$
$$p \mapsto (x^0, x^1, x^2, x^3)$$

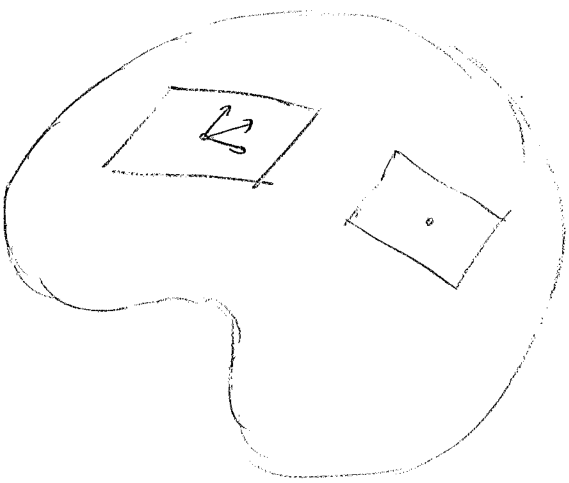
$$\varphi_2 : U_2 \to \tilde{U}_2 \subset \mathbb{R}^4$$
$$p \longmapsto (y^0, y^1, y^2, y^3)$$

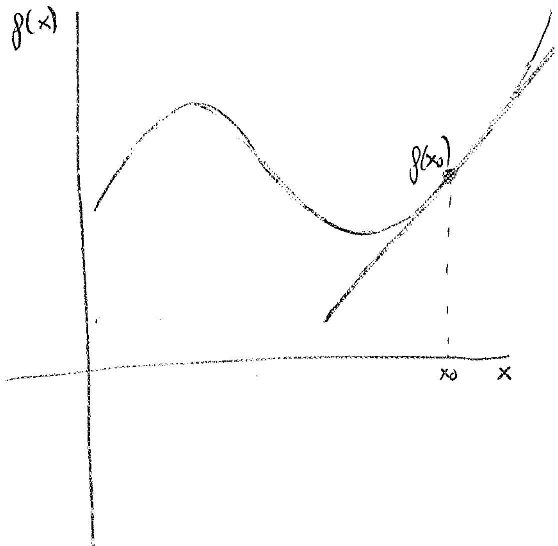
$$\varphi_2 \circ \varphi_1^{-1} : \tilde{U}_2 \to \tilde{U}_1$$
$$(x^0, x^1, x^2, x^3) \mapsto (y^0, y^1, y^2, y^3)$$

$$\begin{cases} y^0 = y^0(x^0, x^1, x^2, x^3) \\ \vdots \\ y^3 = y^3(x^0, x^1, x^2, x^3) \end{cases}$$









Rindler space

(Wald discussion pp149 [Wa84])

$$ds^2 = -x^2 dt^2 + dx^2$$

$$-\infty < t < \infty \;;\quad 0 < x < \infty$$

- Looking for coordinates "geometrically" motivated.

Null geodesics:

$$K^a = (\dot t, \dot x)$$

Null condition:

$$0 = K^a K_a = -x^2 \dot t^2 + \dot x^2 \iff \dot x^2 = x^2 \dot t^2 \qquad \left(\frac{dx}{d\lambda}\right)^2 = x^2 \left(\frac{dt}{d\lambda}\right)^2$$

This implies

$$\left(\frac{dt}{dx}\right)^2 = \frac{1}{x^2} \qquad \frac{dt}{dx} = \pm\frac{1}{x} \qquad \begin{cases} t = \ln x + u \\ t = -\ln x + v \end{cases}$$

We define "null" coordinates:

$$\begin{cases} u = t - \ln x \\ v = t + \ln x \end{cases} \qquad \begin{cases} t = \frac{1}{2}(u+v) \\ \ln x = \frac{1}{2}(v-u) \end{cases} \qquad x = e^{(v-u)/2} \qquad \begin{matrix} -\infty < u < \infty \\ -\infty < v < \infty \end{matrix}$$

$$\begin{cases} du = dt - \frac{1}{x} dx \\ dv = dt + \frac{1}{x} dx \end{cases} \qquad \begin{matrix} dt = \frac{1}{2}(du + dv) \\ \frac{1}{x} dx = \frac{1}{2}(du - dv) \end{matrix}$$

$$dx = \frac{x}{2}(du - dv) = \frac{e^{(v-u)/2}}{2}(du - dv)$$

Then:

$$\boxed{ds^2} = -x^2 dt^2 + dx^2 = -\overbrace{e^{(v-u)}}^{x^2} \overbrace{\frac{1}{4}(du + dv)^2}^{dt^2} + \overbrace{\frac{e^{(v-u)}}{4}(du - dv)^2}^{dx^2} =$$

$$= \frac{e^{(v-u)}}{4}(-)\,4\,du\,dv = \boxed{-e^{v-u} du\,dv}$$

Change of variables to an affine parameter. Note: $K_a t^a$ is a constant of motion for t^a Killing.

Then, since $\left(\frac{\partial}{\partial t}\right)^a$ is a Killing:

$$E = -g_{ab} K^a \left(\frac{\partial}{\partial t}\right)^b = x^2 \frac{dt}{d\lambda}$$

Then, along $u = $ cte (outgoing null rays)

$$E = e^{v-u} \frac{1}{2}\left(\cancel{\frac{du}{d\lambda}} + \frac{dv}{d\lambda}\right) \longrightarrow \lambda = \frac{1}{2E} e^{-u} \int e^{v} = C + \frac{1}{2E} e^{-u} e^{v}$$

Therefore $\lambda_{out} = e^{v}$ is an affine parameter

Analogously, for $v = \text{const}$.

$$E = e^{v-u}\frac{1}{2}\frac{dv}{d\lambda} \qquad \lambda = C + \frac{1}{2E}\, e^{v}\ e^{-u}$$

$$\lambda_{in} = e^{-u}$$

Let us define:

$$U = -e^{-u} \qquad V = e^{v} \qquad 0 < V < \infty$$
$$-\infty < U < 0$$

$$dU = e^{-u} du \ ; \ dV = e^{v} dv$$

$$ds^2 = -e^{v-u} du dv = -e^{v-u} e^{u} dU\, e^{-v} dV = -dUdV$$

Rindler: $ds^2 = -dUdV \qquad U < 0$
$V > 0$

Extended: $ds^2 = -dU dV \qquad -\infty < U < \infty$
$-\infty < V < \infty$

Minkowski: $T = \frac{(U+V)}{2} \qquad X = \frac{(V-U)}{2}$

$$V = T + X \qquad dV = dT + dX$$
$$U = T - X \qquad dU = dT - dX$$

$$\boxed{ds^2} = -(dT - dX)(dT + dX) = \boxed{-dT^2 + dX^2}$$

- Relation (t,x) and (T,X)

$$\begin{cases} x = e^{(v-u)/2} \\ t = \frac{1}{2}(u+v) \end{cases} \qquad \begin{cases} -e^{-u} = U \\ e^{v} = V \end{cases} \qquad \begin{cases} U = T - X \\ V = T + X \end{cases}$$

$$\boxed{x} = (-VU)^{1/2} = (-(T+X)(T-X))^{1/2} = \boxed{(X^2 - T^2)^{1/2}}$$

$$-u = \ln(-U)$$
$$v = \ln V$$

$$\boxed{t} = \frac{1}{2}(-\ln(-U) + \ln V) = \frac{1}{2}\left(\ln\left(\frac{-1}{U}\right) + \ln V\right) = \frac{1}{2}\ln\left(-\frac{V}{U}\right) = \frac{1}{2}\ln\left(\frac{T+X}{-(T-X)}\right) =$$

$$= \frac{1}{2}\ln\left(\frac{X+T}{X-T}\right) = \frac{1}{2}\ln\left(\frac{1+T/X}{1-T/X}\right) = \boxed{\tanh^{-1}(T/X)}$$

$$\tanh^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \qquad \text{(8.57 Spiegel)}$$

$(x,t) \qquad (u,v)$

$$\partial_t = \frac{\partial u}{\partial t}\frac{\partial}{\partial u} + \frac{\partial v}{\partial t}\frac{\partial}{\partial v} =$$

$$= \frac{\partial}{\partial u} + \frac{\partial}{\partial v} = -U\frac{\partial}{\partial U} + V\frac{\partial}{\partial V}$$

$(u,v) \leftrightarrow (U,V)$

$$\frac{\partial}{\partial u} = \frac{\partial U}{\partial u}\frac{\partial}{\partial U} + \frac{\partial V}{\partial u}\frac{\partial}{\partial V} = +e^{-u}\frac{\partial}{\partial U} + 0 = -U\frac{\partial}{\partial U}$$

$$\frac{\partial}{\partial v} = \frac{\partial U}{\partial v}\frac{\partial}{\partial U} + \frac{\partial V}{\partial v}\frac{\partial}{\partial V} = e^{v}\frac{\partial}{\partial V} = V\frac{\partial}{\partial V}$$

$(U,V) \leftrightarrow (T,X) \longrightarrow \quad T = \frac{1}{2}(U+V), \quad X = \frac{1}{2}(V-U)$

$$\frac{\partial}{\partial U} = \frac{\partial T}{\partial U}\frac{\partial}{\partial T} + \frac{\partial X}{\partial U}\frac{\partial}{\partial X} =$$

$$= \frac{1}{2}\frac{\partial}{\partial T} - \frac{1}{2}\frac{\partial}{\partial X}$$

$$\frac{\partial}{\partial V} = \frac{\partial T}{\partial V}\frac{\partial}{\partial T} + \frac{\partial X}{\partial V}\frac{\partial}{\partial X} =$$

$$= \frac{1}{2}\frac{\partial}{\partial T} + \frac{1}{2}\frac{\partial}{\partial X}$$

$$\boxed{\partial_t} = \underbrace{-U\frac{\partial}{\partial U} + V\frac{\partial}{\partial V}}_{\text{dilatační mih}} = \frac{1}{2}\Big((T-X)(\partial_T - \partial_X) + (T+X)(\partial_T + \partial_X)\Big) =$$

$$= \frac{1}{2}\Big[\cancel{-T\partial_T} + T\partial_X + X\partial_T \cancel{-X\partial_X} + \cancel{T\partial_T} + T\partial_X + X\partial_T + \cancel{X\partial_X}\Big] =$$

$$= \boxed{T\partial_X + X\partial_T}$$

- Inverse transformation

$$\begin{cases} x = (X^2 - T^2)^{1/2} \\ t = \tanh^{-1}\left(\frac{T}{X}\right) \end{cases}$$

Note that x and t (or at least t) is dimensionless:

$$x^2 = X^2 - T^2$$

$$1 = \left(\frac{X}{x}\right)^2 - \left(\frac{T}{x}\right)^2$$

Then,

$$\left.\begin{array}{l} \frac{X}{x} = \cosh\alpha \\ \frac{T}{x} = \sinh\alpha \end{array}\right\} \longrightarrow \tanh\alpha = \frac{T}{X} \qquad \alpha = \tanh^{-1}\left(\frac{T}{X}\right) \quad (*)$$

On the other hand, for a fixed x_0 we can introduce a dimensionful t:

$$t \rightarrow \frac{t}{x_0}$$

so that:

$$\frac{t}{x_0} = \tanh^{-1}\left(\frac{T}{X}\right) \qquad (**)$$

From (*) and (**):

$$\alpha = \frac{t}{x_0}$$

Therefore:

$$\begin{cases} X = x\cosh\left(\frac{t}{x_0}\right) \\ T = x\sinh\left(\frac{t}{x_0}\right) \end{cases}$$

For fixed x and $x = x_0$, we have

$$\alpha = \frac{v}{c} = \frac{t}{x} = at \iff \boxed{a = \frac{1}{x}}$$ → relate later to Schwarzschild acceleration of a static observer at the horizon

- Resemblance to Schwarzschild

Let us make the change $y = x^2$ in $ds^2 = -x^2dt^2 + dx^2$.

Then: $dy = 2x\,dx$; $dx = \frac{1}{2x}dy = \frac{1}{2\sqrt{y}}dy$; $dx^2 = \frac{1}{4y}dy^2$

So that:

$$ds^2 = -y\,dt^2 + \frac{1}{4y}dy^2$$

Compare to: $ds^2 = -f\,dt^2 + f^{-1}dr^2 + d\Omega^2$, with $\boxed{f = 1 - \frac{2M}{r}}$

$t = \frac{1}{2}(u+v)$

$u = -\ln(-U) = \ln\left(\frac{-1}{U}\right)$

$v = \ln V$

$u+v = \ln\left(\frac{-1}{U}\right) + \ln(V) = \ln\left(\frac{-V}{U}\right) =$

$= \ln\left(\frac{\cancel{x}(T+x)}{(T-x)}\right) = \ln\left(\frac{x+T}{x-T}\right) = \ln\left(\frac{1+\frac{T}{x}}{1-T/x}\right)$

$\operatorname{senh}(x) = \frac{e^{x} - e^{-x}}{2}$

$\cosh(x) = \frac{e^{x}+e^{-x}}{2}$

$\tanh(x) = \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}$

$\tanh^{-1}(x) = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right)$

$\partial_t = \frac{\partial T}{\partial t}\frac{\partial}{\partial T} + \frac{\partial X}{\partial t}\frac{\partial}{\partial X} =$

$$\begin{cases} x = (X^2 - T^2)^{1/2} \\ t = \tanh^{-1}(T/x) \end{cases}$$

$\frac{T}{X} = \tanh t$

$\frac{\partial x}{\partial t} = \frac{1}{2}\left(2X\frac{\partial X}{\partial t} - 2T\frac{\partial T}{\partial t}\right)(X^2-T^2)^{-1/2} =$

$= \left(X\frac{\partial X}{\partial t} - T\frac{\partial T}{\partial t}\right)(X^2-T^2)^{-1/2}$

$\frac{\partial t}{\partial X} =$

$\frac{d}{dx}(f^{-1})(x) = \frac{1}{f(x)}$

$f \circ f^{-1} = I$

$\frac{d}{dx} f(f^{-1}) \cdot \frac{d}{d} f^{-1} = 1$

$f \frac{df^{-1}}{dx} = \frac{1}{f'}$

1.3. Covariant derivative

Given the scalar $f: M \to \mathbb{R}$ we can define a $\binom{0}{1}$ tensor (the gradient) by considering in some coordinate system:

$$\nabla f = \nabla_\mu f \, dx^\mu$$

$$\nabla_\mu f = \frac{\partial f}{\partial x^\mu}$$

Indeed $\nabla_\mu f$ transforms as it should:

$$\nabla_{\mu'} f = \frac{\partial f}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial f}{\partial x^\nu} = \left(\frac{\partial x^\nu}{\partial x'^\mu}\right)\nabla_\nu f \quad \text{Ok}$$

Let us consider now a vector $V = V^\mu \partial_\mu$ and let us try to define the gradient as:

$$\nabla_\mu V^\nu = \partial_\mu V^\nu$$

Then we find:

$$\nabla_{\mu'} V'^\nu = \frac{\partial}{\partial x'^\mu}\left(\left(\frac{\partial x'^\nu}{\partial x^\rho}\right)V^\rho\right) = \left(\frac{\partial x^\sigma}{\partial x'^\mu}\right)\frac{\partial}{\partial x^\sigma}\left(\left(\frac{\partial x'^\nu}{\partial x^\rho}\right)V^\rho\right) =$$

$$= \left(\frac{\partial x^\sigma}{\partial x'^\mu}\right)\left(\frac{\partial x'^\nu}{\partial x^\rho}\right)\frac{\partial V^\rho}{\partial x^\sigma} + \underbrace{\left(\frac{\partial x^\sigma}{\partial x'^\mu}\right)\frac{\partial^2 x'^\nu}{\partial x^\sigma \partial x^\rho} V^\rho}_{\text{non-tensorial part.}}$$

In order to solve this we introduce a new derivative operator:

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\rho} V^\rho$$

Such that:

$$\nabla_{\mu'} V'^\nu = \nabla_\mu V^\nu$$

First:

$$\nabla_{\mu'} V'^\nu = \partial_{\mu'} V'^\nu + \Gamma'^\nu_{\mu\sigma} V'^\sigma = \left(\frac{\partial x^\sigma}{\partial x'^\mu}\right)\left(\frac{\partial x'^\nu}{\partial x^\rho}\right)\frac{\partial V^\rho}{\partial x^\sigma} + \left(\frac{\partial x^\sigma}{\partial x'^\mu}\right)\frac{\partial^2 x'^\nu}{\partial x^\sigma \partial x^\rho} V^\rho$$

$$+ \Gamma'^\nu_{\mu\sigma}\left(\frac{\partial x'^\sigma}{\partial x^\rho}\right)V^\rho$$

We want,

$$\nabla_{\mu'} V'^\nu = \left(\frac{\partial x^\sigma}{\partial x'^\mu}\right)\left(\frac{\partial x'^\nu}{\partial x^\rho}\right)\nabla_\sigma V^\rho = \left(\frac{\partial x^\sigma}{\partial x'^\mu}\right)\left(\frac{\partial x'^\nu}{\partial x^\rho}\right)(\partial_\sigma V^\rho + \Gamma^\rho_{\sigma\lambda} V^\lambda) =$$

$$= (\ )(\ )\partial_\sigma V^\rho + \underbrace{\left(\frac{\partial x^\sigma}{\partial x'^\mu}\right)\left(\frac{\partial x'^\nu}{\partial x^\rho}\right)\Gamma^\rho_{\sigma\lambda} V^\lambda}_{\left(\frac{\partial x^\sigma}{\partial x'^\mu}\right)\left(\frac{\partial x'^\nu}{\partial x^\lambda}\right)\Gamma^\lambda_{\sigma\rho} V^\rho}$$

So we can write:

$$V^\rho\left(\Gamma'^\nu_{\mu\sigma}\left(\frac{\partial x'^\sigma}{\partial x^\rho}\right)+\frac{\partial x^\sigma}{\partial x'^\mu}\frac{\partial^2 x'^\nu}{\partial x^\sigma\partial x^\rho}\right)=\Gamma^\lambda_{\sigma\rho}\left(\frac{\partial x^\sigma}{\partial x'^\mu}\right)\left(\frac{\partial x'^\nu}{\partial x^\lambda}\right)V^\rho \quad , \forall V$$

$$\Gamma'^\nu_{\mu\sigma}\left(\frac{\partial x'^\sigma}{\partial x^\rho}\right)=\left(\frac{\partial x^\sigma}{\partial x'^\mu}\right)\left(\frac{\partial x'^\nu}{\partial x^\lambda}\right)\Gamma^\lambda_{\sigma\rho}-\left(\frac{\partial x^\sigma}{\partial x'^\mu}\right)\frac{\partial^2 x'^\nu}{\partial x^\sigma\partial x^\rho}$$

Multiply by $\left(\frac{\partial x^\rho}{\partial x'^{\tilde\rho}}\right)$:

$$\Gamma'^\nu_{\mu\sigma}\,\delta^\sigma_{\tilde\rho}=\left(\frac{\partial x^\rho}{\partial x'^{\tilde\rho}}\right)\left(\frac{\partial x^\sigma}{\partial x'^\mu}\right)\left(\frac{\partial x'^\nu}{\partial x^\lambda}\right)\Gamma^\lambda_{\sigma\rho}-\left(\frac{\partial x^\rho}{\partial x'^{\tilde\rho}}\right)\left(\frac{\partial x^\sigma}{\partial x'^\mu}\right)\frac{\partial^2 x'^\nu}{\partial x^\sigma\partial x^\rho}$$

$$\boxed{\Gamma'^\nu_{\mu\gamma}=\left(\frac{\partial x^\sigma}{\partial x'^\mu}\right)\left(\frac{\partial x^\rho}{\partial x'^\gamma}\right)\left(\frac{\partial x'^\nu}{\partial x^\lambda}\right)\Gamma^\lambda_{\sigma\rho}-\underbrace{\left(\frac{\partial x^\rho}{\partial x'^\gamma}\right)\left(\frac{\partial x^\sigma}{\partial x'^\mu}\right)\frac{\partial^2 x'^\nu}{\partial x^\sigma\partial x^\rho}}_{\text{not a tensor.}}}$$

reconciliate with (3.3.3) in FroZel.

- $D_\mu \alpha_\nu = \partial_\mu \alpha_\nu - \Gamma^\rho_{\mu\nu}\alpha_\rho$
- Parallelism along a curve $u^\mu = \frac{dx^\mu}{d\lambda}$
- Levi-Civita connect.: How we get a $\Gamma^\mu_{\nu\rho}$ with such properties: metric $g_{\mu\nu}$
  - Torsion-fr

    We want: $D_\mu D_\nu f = D_\nu D_\mu f$ (torsion-free)

    $$\left.\begin{aligned} D_\mu D_\nu f &= D_\mu \partial_\nu f = \partial_\mu\partial_\nu f - \Gamma^\rho_{\mu\nu}\partial_\rho f \\ D_\nu D_\mu f &= D_\nu \partial_\mu f = \partial_\nu\partial_\mu f - \Gamma^\rho_{\nu\mu}\partial_\rho f \end{aligned}\right\} \Longrightarrow \boxed{\Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\nu\mu}}$$

  - Compatible with the metric: $D_\mu g_{\nu\rho} = 0$

    $$0 = D_\mu g_{\nu\rho} = \partial_\mu g_{\nu\rho} - \Gamma^\sigma_{\mu\nu} g_{\sigma\rho} - \Gamma^\sigma_{\mu\rho} g_{\nu\sigma}$$

    So,

    $$\partial_\mu g_{\nu\rho} = \Gamma^\sigma_{\mu\nu} g_{\sigma\rho} + \Gamma^\sigma_{\mu\rho} g_{\nu\sigma} = \Gamma_{\rho\mu\nu} + \Gamma_{\nu\mu\rho} \quad (*)$$

    Permute indices:

    $$\partial_\nu g_{\mu\rho} = \Gamma_{\rho\nu\mu} + \Gamma_{\mu\nu\rho} \quad (**)$$

    $$\partial_\rho g_{\mu\nu} = \Gamma_{\nu\rho\mu} + \Gamma_{\mu\rho\nu} \quad (***)$$

Then (*) + (**) − (***)

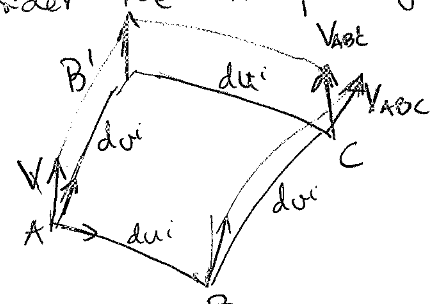
$$\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu} = \underset{\substack{\uparrow \\ \text{symmetry}}}{\Gamma_{\rho\mu\nu}} + \cancel{\Gamma_{\nu\mu\rho}} + \Gamma_{\rho\nu\mu} + \cancel{\Gamma_{\mu\nu\rho}} - \cancel{\Gamma_{\nu\rho\mu}} - \cancel{\Gamma_{\mu\rho\nu}} = 2\Gamma_{\rho\mu\nu}$$

$$\boxed{\Gamma^{\rho}{}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right)}$$

This $\Gamma^{\rho}_{\mu\nu}$ transforms as it must.

1.4. Curvature

Let us consider the transport of a vector V^i



$$\boxed{V_{ABC} - V_{AB'C} = \underset{\substack{\uparrow \\ \text{curvature}}}{R}\, V}$$

$A \to B$: $\quad V^\mu \to V^\mu + du^\nu D_\nu V^\mu$

$B \to C$: $\quad (V^\mu + du^\nu D_\nu V^\mu) + dv^\rho D_\rho (V^\mu + du^\nu D_\nu V^\mu) =$

$$= V^\mu + du^\nu D_\nu V^\mu + dv^\rho D_\rho V^\mu + du^\nu dv^\rho D_\rho D_\nu V^\mu \quad (*)$$

$A \to B' =$ $\quad V^\mu + dv^\nu D_\nu V^\mu$

$B' \to C =$ $\quad (V^\mu + dv^\nu D_\nu V^\mu) + du^\rho D_\rho (V^\mu + dv^\nu D_\nu V^\mu) =$

$$= V^\mu + dv^\nu D_\nu V^\mu + du^\nu D_\nu V^\mu + \underbrace{du^\rho dv^\nu D_\rho D_\nu V^\mu}_{du^\nu dv^\rho D_\nu D_\rho V^\mu} \quad (**)$$

Then: (**) − (*):

$$du^\nu dv^\rho (D_\nu D_\rho - D_\rho D_\nu) V^\mu \sim \underbrace{R^\mu{}_{\lambda\nu\rho}}_{\text{curvature}} V^\lambda$$

Then:

$$\nabla_\nu \nabla_\rho V^\mu = \partial_\nu(\nabla_\rho V^\mu) - \Gamma^\lambda_{\nu\rho}\nabla_\lambda V^\mu + \Gamma^\mu_{\nu\lambda}\nabla_\rho V^\lambda =$$

$$= \partial_\nu(\partial_\rho V^\mu + \Gamma^\mu_{\rho\sigma}V^\sigma) - \Gamma^\lambda_{\nu\rho}(\partial_\lambda V^\mu + \Gamma^\mu_{\lambda\sigma}V^\sigma) +$$

$$+ \Gamma^\mu_{\nu\lambda}(\partial_\rho V^\lambda + \Gamma^\lambda_{\rho\sigma}V^\sigma) =$$

$$= \partial_\nu\partial_\rho V^\mu + \partial_\nu\Gamma^\mu_{\rho\sigma}V^\sigma + \Gamma^\mu_{\rho\sigma}\partial_\nu V^\sigma - \Gamma^\lambda_{\nu\rho}\partial_\lambda V^\mu - \Gamma^\lambda_{\nu\rho}\Gamma^\mu_{\lambda\sigma}V^\sigma +$$

$$+ \Gamma^\mu_{\nu\lambda}\partial_\rho V^\lambda + \Gamma^\mu_{\nu\lambda}\Gamma^\lambda_{\rho\sigma}V^\sigma \quad (*)$$

$$\nabla_\rho\nabla_\nu V^\mu = \partial_\rho\partial_\nu V^\mu + \partial_\rho\Gamma^\mu_{\nu\sigma}V^\sigma + \Gamma^\mu_{\nu\sigma}\partial_\rho V^\sigma - \Gamma^\lambda_{\rho\nu}\partial_\lambda V^\mu - \Gamma^\lambda_{\rho\nu}\Gamma^\mu_{\lambda\sigma}V^\sigma +$$

$$+ \Gamma^\mu_{\rho\lambda}\partial_\nu V^\lambda + \Gamma^\mu_{\rho\lambda}\Gamma^\lambda_{\nu\sigma}V^\sigma \quad (**)$$

$(*) - (**)$:

$$(\nabla_\nu\nabla_\rho - \nabla_\rho\nabla_\nu)V^\mu = \cancel{\partial_\nu\partial_\rho V^\mu} + \underline{\partial_\nu\Gamma^\mu_{\rho\sigma}V^\sigma} + \cancel{\Gamma^\mu_{\rho\sigma}\partial_\nu V^\sigma} - \cancel{\Gamma^\lambda_{\nu\rho}\partial_\lambda V^\mu} - \Gamma^\lambda_{\nu\rho}\Gamma^\mu_{\lambda\sigma}V^\sigma$$

$$+ \cancel{\Gamma^\mu_{\nu\lambda}\partial_\rho V^\lambda} + \Gamma^\mu_{\nu\lambda}\Gamma^\lambda_{\rho\sigma}V^\sigma$$

$$- \cancel{\partial_\rho\partial_\nu V^\mu} - \underline{\partial_\rho\Gamma^\mu_{\nu\sigma}V^\sigma} - \cancel{\Gamma^\mu_{\nu\sigma}\partial_\rho V^\sigma} + \cancel{\Gamma^\lambda_{\rho\nu}\partial_\lambda V^\mu} + \underline{\Gamma^\lambda_{\rho\nu}\Gamma^\mu_{\lambda\sigma}V^\sigma} -$$

$$- \cancel{\Gamma^\mu_{\rho\lambda}\partial_\nu V^\lambda} - \Gamma^\mu_{\rho\lambda}\Gamma^\lambda_{\sigma\sigma}V^\sigma =$$

$$= \Big(\underbrace{\partial_\nu\Gamma^\mu_{\rho\sigma} - \partial_\rho\Gamma^\mu_{\nu\sigma} + \Gamma^\lambda_{\rho\nu}\Gamma^\mu_{\lambda\sigma} - \Gamma^\lambda_{\nu\rho}\Gamma^\mu_{\lambda\sigma}}_{R^\mu{}_{\lambda\nu\rho}}\Big) V^\sigma$$

- Ricci tensor, Ricci scalar   Bianchi identities
- Weyl tensor
-

Derivation of Einstein equation from a Lagrangian perspective

1). Einstein - Hilbert action

• **Measure**: Given a scalar field "f", we cannot give a sense to

$$\int f \, d^4x$$

because this depends on the chosen coordinate system. $d^4x \longrightarrow \left|\frac{\partial x'}{\partial x}\right| d^4x$

We need a measure well defined on the spacetime. This is essentially a n-form, i.e. an $\binom{0}{n}$ tensor antisymmetric.

We avoid introducing forms.

We notice that given a metric $g_{\mu\nu}$, we can define the measure $d\mu_g$, that in a coordinate system $\{x^\mu\}$ writes

$$\mu_g = \sqrt{|g|}\, d^4x$$

with $g = \det(g_{ab})$.

Then, using the tensorial change to $\{x'^\mu\}$

$$g'_{\mu\nu} = \left(\frac{\partial x^\rho}{\partial x'^\mu}\right)\left(\frac{\partial x^\sigma}{\partial x'^\nu}\right) g_{\rho\sigma}$$

and $\det(AB) = \det A \det B$, we have.

$$g' = \left|\left(\frac{\partial x}{\partial x'}\right)\right|^2 g$$

so

$$\sqrt{g'} = \left|\left(\frac{\partial x}{\partial x'}\right)\right|\sqrt{g}$$

On the other hand, under a change of variables $\{x^\mu\} \to \{x'^\mu\}$ a multiple integral transform with the jacobian of the transformation:

$$d^4x \longrightarrow \left|\left(\frac{\partial x'}{\partial x}\right)\right| d^4x$$

Therefore:

$$\sqrt{g}\, d^4x = \sqrt{g'}\, d^4x'$$

and we can integrate:

$$\int f \sqrt{g}\, d^4x$$

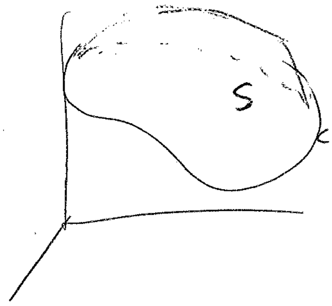
Stokes theorem.

We are familiar with

$$F(b)-F(a) = \int \frac{d}{dt} f \, dt$$ (Fundamental theory of calculus) with F the primitive

or the Stokes (Green) theorems

$$\int (\nabla \times A) \cdot d\vec{S} = \oint_C A \cdot d\vec{r}$$



or Gauss

$$\int \nabla \cdot A \, dV = \int A \cdot d\vec{S}$$

in general $\int_M d\alpha = \int_{\partial M} \alpha$

In our case we need a version:

$$\boxed{\int_M \nabla_\mu v^\mu \, d\mu_g = \int_{\partial M} v^\mu \cdot s_\mu \, d\mu_{\tilde{g}}}$$, where $\tilde{g}$ is the induced metric on ∂M

- Choice of scalar, out of the metric: Not much from where we can choose. Contractions of the Riemann tensor.

  The simplest R scalar $\longrightarrow f(R)$

  Simplest $f(R) = R$.

- B-F theory in 4d.

  B two form. A gauge connecti

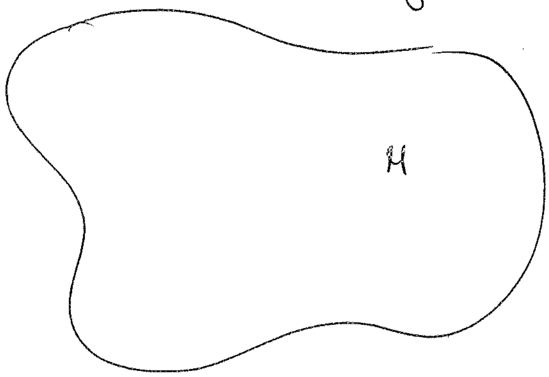
  $F = dA + A \wedge A$

  $S \sim \int B \wedge F$

• Einstein Hilbert action

$$S = \int R\sqrt{-g}\, d^4x$$

Let us consider the 2nd order variational problem on a domain M, where the variable to be varied is g_{ab}: Other options (1st order variational Palatini action.



such that we fix on the boundary δg_{ab}.

Elements:

• Metric g_{ab}

• Christoffel symbols

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}\left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right) = \frac{1}{2} g^{\rho\sigma}\left(g_{\sigma\nu,\mu} + g_{\sigma\mu,\nu} - g_{\mu\nu,\sigma}\right)$$

• Ricci curvature

$$R_{\mu\nu} = \Gamma^\rho_{\mu\nu,\rho} - \Gamma^\rho_{\mu\rho,\nu} + \Gamma^\rho_{\sigma\rho}\Gamma^\sigma_{\mu\nu} - \Gamma^\sigma_{\rho\nu}\Gamma^\rho_{\mu\sigma}$$

• Useful relations:

• $g^{\mu\rho} g_{\nu\rho} = \delta^\mu{}_\nu \rightarrow (\delta g^{\mu\rho})\, g_{\nu\rho} + g^{\mu\rho}\,\delta g_{\nu\rho} = 0$

$$g_{\rho\mu} g_{\sigma\nu}\,\delta g^{\rho\sigma} = -\underbrace{g_{\rho\mu} g^{\rho\sigma}}_{\delta_\mu{}^\sigma}\,\delta g_{\sigma\nu} = -\delta g_{\mu\nu}$$

$$\delta \ln|g| = g^{\mu\nu}\delta g_{\mu\nu}$$

$\delta\sqrt{-g}$: $\quad \delta \ln\sqrt{g} = \dfrac{\delta\sqrt{g}}{\sqrt{g}}$

$$\left.\begin{array}{l} \delta \ln\sqrt{g} = \dfrac{\delta\sqrt{g}}{\sqrt{g}} \\ \delta \ln\sqrt{g} = \frac{1}{2}\delta \ln g = \frac{1}{2} g^{\mu\nu}\delta g_{\mu\nu} \end{array}\right\} \delta\sqrt{g} = \frac{1}{2}\sqrt{g}\, g^{\mu\nu}\,\delta g_{\mu\nu} = -\frac{1}{2}\sqrt{g}\, g_{\mu\nu}\,\delta g^{\mu\nu}$$

. Then we have

$$\delta S = \delta \int_M R\sqrt{-g}\, d^4x = \delta\int_M g^{\mu\nu}R_{\mu\nu}\sqrt{-g}\, d^4x =$$

$$= \int\left(\delta g^{\mu\nu} R_{\mu\nu}\sqrt{-g} + g^{\mu\nu}\delta R_{\mu\nu}\sqrt{-g} + g^{\mu\nu}R_{\mu\nu}\,\delta\sqrt{-g}\right) d^4x =$$

Try to write as $(\dots)\,\delta g_{\mu\nu}\sqrt{-g}$ for arbitrary $\delta g^{\mu\nu}$

. First: $\quad \delta g_{\mu\nu} R^{\mu\nu}\sqrt{-g} = R_{\mu\nu}\,\delta g^{\mu\nu}\sqrt{-g}$

. Third:

$$g^{\mu\nu}R_{\mu\nu}\,\delta\sqrt{-g} = g^{\rho\sigma}R_{\rho\sigma}\underbrace{\frac{(-1)}{2} g_{\mu\nu}\,\delta g^{\mu\nu}\sqrt{-g}}_{\delta\sqrt{g}} =$$

$$= -\frac{1}{2} R\, g_{\mu\nu}\,\delta g^{\mu\nu}\sqrt{-g}$$

. Second:

Choose coordinates where $\Gamma^{\mu}_{\nu\rho} = 0$

Then:

$$\delta R_{ab} \overset{\Gamma = 0}{=} \delta\left(\Gamma^{c}{}_{ab,c}\right) - \delta\left(\Gamma^{c}{}_{ac,b}\right)$$

$$\overset{(\dots)_{,c} = \partial_c(\dots)}{=} (\delta\Gamma^{c}{}_{ab})_{,c} - (\delta\Gamma^{c}{}_{ac})_{,b} =$$

$$\overset{\Gamma^{a}_{ab} = 0}{=} (\delta\Gamma^{c}{}_{ab})_{;c} - (\delta\Gamma^{c}{}_{ac})_{;b} \quad \text{(tensorial!!)}$$

Then.

$$g^{\mu\nu}\,\delta R_{\mu\nu} \overset{g_{\mu\nu;\sigma}=0}{=} (g^{\mu\nu}\delta\Gamma^{\rho}_{\mu\nu})_{;\rho} - (g^{\mu\nu}\delta\Gamma^{\rho}_{\mu\rho})_{;\nu}$$

$$= \underbrace{(g^{\mu\nu}\,\delta\Gamma^{\rho}_{\mu\nu} - g^{\mu\rho}\,\delta\Gamma^{\nu}_{\mu\nu})}_{v^\rho}{}_{;\rho}$$

That is, $\quad g^{\mu\nu}\delta R_{\mu\nu} = v^{\rho}{}_{;\rho} = \nabla_\rho v^\rho$

It can be seen:

$$v_a = \nabla^b(\delta g_{ab}) - g^{cd}\nabla_a(\delta g_{cd})$$ using expression of $\Gamma = g^{..}(\partial ..)$!

Therefore:

$$\delta S = \int_M \Big(\underbrace{R_{\mu\nu}}_{1^{st}} - \underbrace{\tfrac{1}{2} R g_{\mu\nu}}_{3^{rd}}\Big)\,\delta g^{\mu\nu}\sqrt{-g} + \underbrace{\int_M \nabla_\rho v^\rho}$$

$$\int_{\partial M} v^\rho s_\rho \sqrt{g}\, d^4x = 0$$

$$\downarrow$$

$$v^\rho \sim \delta g_{\mu\nu}\big|_{\partial M} = 0$$

This holds $\forall\, g_{\mu\nu}$, therefore:

$$\boxed{G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} = 0}$$

- Matter

Consider a matter Lagrangian

$$S_M = \int_M \mathcal{L}_M(\psi, \partial\psi)\sqrt{g}\, d^4x$$

Equations of motion

$$\frac{\delta S_M}{\delta \psi} = 0$$

$$\delta S_M = \int \frac{\delta S_M}{\delta \psi}\,\delta\psi \qquad \delta S = \int \frac{\delta \mathcal{L}_M}{\delta \psi}\,\delta\psi\,\sqrt{g}\, d^4x \;\overset{\forall \delta\psi}{\Longrightarrow}\; \frac{\delta \mathcal{L}_M}{\delta \psi}\Big|_{\psi} = 0$$

with fixed boundaries $\delta\psi$.

Define now the matter stress-energy tensor as:

$$T_{\mu\nu} = \frac{-1}{8\pi}\frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}} \qquad \left(\delta_g S_M \equiv \int \left(\frac{\delta S_M}{\delta g^{\mu\nu}}\right)\delta g^{\mu\nu}\right)$$

(symmetric!!)

This define $\frac{\delta S_M}{\delta g^{\mu\nu}}$

Then, writing

$$S = S_H + S_M$$

the matter equations are the same ($S_H \neq S_M(\psi)$), but the Einstein eq. is modified:

$$\delta S = \int \left(R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu}\right)\delta g^{\mu\nu}\sqrt{-g} + \int \underbrace{\frac{\delta S_M}{\delta g^{\mu\nu}}}_{= 8\pi\sqrt{-g}\,T_{\mu\nu}\,\delta g^{\mu\nu}} \delta g^{\mu\nu} + \text{b.t.} =$$

$\forall\, g^{\mu\nu}$, therefore:

$$\boxed{R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = 8\pi T_{\mu\nu}}$$

- Conservation of $T_{\mu\nu}$ under matter equation eqs. S_M scalar $\to$ invariant under diffeos

$$0 = \delta_\xi S_M = \int \frac{\delta S_M}{\delta g^{\mu\nu}}\delta_\xi g^{\mu\nu} + \overset{0 \text{ e.o.m}}{\cancel{\frac{\delta S_M}{\delta \psi}}}\delta_\xi \psi =$$

$$\delta_\xi g^{\mu\nu} \stackrel{*}{=} \mathcal{L}_\xi g^{\mu\nu} = \nabla^\mu \xi^\nu + \nabla^\nu \xi^\mu$$

$$\underset{\overset{\uparrow}{\text{dg } T_{\mu\nu}}}{0} = \int \sqrt{-g}\,T_{\mu\nu}\,\delta_\xi g^{\mu\nu} = \int \sqrt{-g}\,T_{\mu\nu}\,\mathcal{L}_\xi g^{\mu\nu}\, d^4x$$

$$= \int \sqrt{-g}\, T_{\mu\nu}(\nabla^\mu \xi^\nu + \nabla^\nu \xi^\mu)\, d^4x =$$

$$= 2\int \sqrt{-g}\, T_{\mu\nu}\nabla^\mu \xi^\nu d^4x = \qquad + \text{b.t.}$$

$$= -2\int \sqrt{-g}\,(\nabla_\mu T^{\mu\nu})\,\xi^\nu\, d^4x \qquad \forall \xi \Rightarrow$$

$$\Rightarrow \boxed{\nabla^\mu T_{\mu\nu} = 0} \quad \text{under b.t.}$$

Schwarzschild construction

- Static and spherically symmetric spacetime

  - Stationary: t^μ timelike killing vector
  - Static: t^μ orthogonal to surfaces Σ (doesn't twist)

  In that case:

$$ds^2 = -V^2 dt^2 + h_{ij} dx^i dx^j$$

$$V \neq V(t)$$
$$h_{ij} \neq h_{ij}(t)$$

- Spherically symmetric
  Invariant under $SO(3)$
  - Spheres S: orbits of this action
  - Induced metric $\rightarrow$ round spheres
    $4\pi r^2 = A \qquad r = \sqrt{\frac{A}{4\pi}}$

$$ds^2 = r^2 (d\theta^2 + \sin^2\theta \, d\varphi^2)$$

- Static and spherical symmetric $\rightarrow$ t^μ orthogonal to S: if it had a projection it would not be $SO(3)$ invariant

  Start from (θ, φ) and
  Therefore orbits S in Σ.
  $\rightarrow$ Choose coordinates (r, θ, φ) in Σ

  (actually stationary + spherically sym $\Rightarrow$ static)
  $0 = t^\mu D_\mu r \Rightarrow t^\mu$ orthogonal to the surface constructed
  geometric — for $D^\mu r$ from S

$$ds^2 = -f(r)\, dt^2 + h(r)\, dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\varphi^2)$$

1) Calculate the Ricci tensor. All components

$$\begin{cases} R_{tt} = \dfrac{-f'(rh f' + f(-4h + rh') + 2rfhf'')}{4rfh^2} \\ R_{rr} = \dfrac{f(4f + rf')h' + rh(f'^2 - 2ff'')}{4rf^2h} \\ R_{\theta\theta} = \dfrac{1}{2}\left(2 - \dfrac{2 + \frac{rf'}{f}}{h} + \dfrac{rh'}{h^2}\right) \\ R_{\varphi\varphi} = \dfrac{\operatorname{sen}^2\theta\,[-rhf' + f(2(-1+h)h + rh')]}{2r^2f^2h^2} \end{cases}$$

We manipulate.

$$E1 = R_{tt}\,4rfh^2 + R_{rr}\,4rf^2h = 4f(hf' + fh')$$

$$E2 = R_{\theta\theta}\,2fh^2 = R_{\varphi\varphi}\frac{2fh^2}{\operatorname{sen}^2\theta} =$$

$$= -rhf' + f(2(-1+h))h + rh')$$

Then

$e_i = 0$ with $f \neq 0$

$$hf' + fh' = 0 \qquad \frac{f'}{f} = -\frac{h'}{h}$$

$$\ln f = \ln k + \ln h^{-1} = \ln(kh^{-1})$$

$$f = kh^{-1}$$

We can set $k = 1$ (it is a rescaling of $t \to k^{1/2} t$)

Sustituting this into $E2 = 0$ leads to:

$$0 = -\frac{2(-1 + f + rf')}{f}$$

Since $f \neq 0$

We have:

$$f'r + f = 1$$

From this

$$f = 1 + \frac{c}{r}$$

- Let us consider the geodesics in this metric and compare them to Newton at large distances $r \to \infty$.

1) First $u^\mu = \frac{dx^\mu}{d\tau} = (\dot{t}, \dot{r}, \dot{\theta}, \dot{\varphi})$

We fix plane $\dot{\theta} = \pi/2$

2) Then

$$-1 = g_{\mu\nu} u^\mu u^\nu = -f \dot{t}^2 + \frac{1}{f}\dot{r}^2 + r^2 \underbrace{\sin^2\theta}_{1} \dot{\varphi}^2 \qquad (*)$$

3) Use that for a Killing vector K^μ, and a geodesic u^μ, the quantity

$K = K^\mu u_\mu$ is preserved along geodesics:

$$u^\mu \nabla_\mu K = u^\mu \nabla_\mu (K_\nu u^\nu) = \underbrace{u^\nu (\nabla_\mu K_\nu) u^\mu}_{\overset{\shortparallel}{0}\ \text{(Killing)}} + K_\nu \underbrace{u^\mu \nabla_\mu u^\nu}_{0} = 0$$

Then: $K^\mu = (\partial_t)^\mu$, $\varphi^\mu = (\partial_\varphi)^\mu$

$$E = -K^\mu u_\mu = f\dot{t} \qquad \dot{t} = -\frac{E}{f}$$

$$L = \varphi^\mu u_\mu = r^2 \dot{\varphi} \qquad \dot{\varphi} = \frac{L}{r^2}$$

Substituting E and L into $(*)$:

$$-1 = -f\frac{E^2}{f^2} + \frac{1}{f}\dot{r}^2 + r^2\frac{L^2}{r^4} = -\frac{E^2}{f} + \frac{1}{f}\dot{r}^2 + \frac{L^2}{r^2}$$

$$E^2 = \dot{r}^2 + f\left(1 + \frac{L^2}{r^2}\right)$$

Let us focus on radial geodesics, $L=0$

$$E^2 = \dot{r}^2 + f(r)$$

$$\dot{r} = \frac{dr}{d\tau} = \frac{dr}{dt}\frac{dt}{d\tau} = \dot{t}\frac{dr}{dt} = -\frac{E}{f}\frac{dr}{dt}$$

Deriving: $\quad f = 1 + \frac{c}{r} \quad \frac{d}{dr}f' = -\frac{c}{r^2}$

$$0 = 2\dot{r}\ddot{r} + \dot{r}\frac{df}{dr} \overset{\dot{r}\neq 0}{\Longleftrightarrow} 2\ddot{r} + \frac{df}{dr} = 0 \Leftarrow 2\ddot{r} + f' = 0 \ ; \ \ddot{r} = -\frac{f'}{2}$$

$$\left(\begin{aligned} \ddot{r} &= \frac{d}{d\tau^2}(r) = \frac{d}{d\tau}\left(\frac{dr}{d\tau}\right) = \frac{d}{d\tau}\left(\underbrace{\frac{dt}{d\tau}}_{\dot{t}}\frac{dr}{dt}\right) = \frac{d}{d\tau}\left(\frac{-E}{f}\frac{dr}{dt}\right) = \underbrace{\frac{dt}{d\tau}}_{\dot{t}}\frac{d}{dt}\left(-\frac{E}{f}\frac{dr}{dt}\right) = \\ &= -E\left(\frac{-E}{f}\right)\frac{d}{dt}\left(\frac{1}{f}\frac{dr}{dt}\right) = \frac{E^2}{f}\left(\frac{dr}{dt}\frac{-f'}{f^2}\frac{dr}{dt} + \frac{1}{f}\frac{d^2r}{dt^2}\right) = \\ &= \frac{E^2}{f}\left(-\frac{f'}{f^2}\left(\frac{dr}{dt}\right)^2 + \frac{1}{f}\frac{d^2r}{dt^2}\right) \end{aligned} \right)$$

For $r \to \infty \qquad f \to 1 \qquad f' = 0$

$$\ddot{r} = \frac{dr}{d\tau^2} \sim \frac{d^2r}{dt^2}$$

$$\left.\begin{aligned} \frac{d^2r}{dt^2} \sim \frac{d^2r}{d\tau^2} &= -\frac{1}{2}f' = \frac{c}{2r^2} \\ \frac{dr^2}{dt^2} &= -\frac{GM}{r^2} \qquad G = 1 \end{aligned}\right\} \quad \frac{c}{2} = -M \quad \boxed{c = -2M}$$

$$\boxed{f = 1 + \frac{c}{r} = 1 - \frac{2M}{r}}$$

Kruskal extension

Spherical symmetry $\to$ 2D "r-t part"

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 \qquad r>0,\ -\infty<t<\infty$$

"singular" in $r=2M$;

Repeat the approach for Rindler:

i) Null coordinates:

$$k^a = (\dot{t}, \dot{x})$$

$$0 = k^a k_a = -\left(1-\frac{2M}{r}\right)\dot{t}^2 + \left(1-\frac{2M}{r}\right)^{-1}\dot{r}^2$$

so that

$$\left(\frac{dt}{dr}\right)^2 = \left(1-\frac{2M}{r}\right)^{-2} = \left(\frac{r}{r-2M}\right)^2$$

This is solved by: $\quad \frac{dt}{dr} = \pm\frac{r}{r-2M} = \pm\frac{1}{1-\frac{2M}{r}} = \pm f^{-1}$

$$t = \pm r_* + C$$

with the tortoise coordinate:

$$r_* = r + 2M\ln\left(\frac{r}{2M}-1\right) \qquad \boxed{\frac{dr_*}{dr} = f^{-1} = \frac{1}{1-\frac{2M}{r}}}$$

That is

$$t = r_* + u$$

$$t = -r_* + v$$

so that we define:

$$\begin{cases} u = t - r_* \\ v = t + r_* \end{cases} \qquad \begin{cases} t = \frac{1}{2}(u+v) \\ r_* = \frac{1}{2}(v-u) \end{cases} \qquad \begin{matrix} -\infty<u<\infty \\ -\infty<v<-\infty \end{matrix}$$

$$r\to 2M;\quad u\to\infty,\ v\to-\infty$$

Therefore:

$$du = dt - dr^* = dt - \frac{dr^*}{dr}dr = dt - f^{-1}dr$$

$$dv = dt + dr^* = dt + \frac{dr^*}{dr}dr = dt + f^{-1}dr$$

$$dt = \frac{1}{2}(du+dv)$$

$$dr = \frac{dr}{dr^*}dr^* = \frac{f}{2}(dv-du)$$

Substituting in $ds^2 = -f\, dt^2 + f^{-1} dr^2$

$$ds^2 = -f\, dt^2 + f^{-1} dr^2 = -f \frac{1}{4}(du + dv)^2 + f^{-1} \frac{f^2}{4}(dv - du)^2 =$$

$$= -f\, du\, dv$$

We rewrite $f = 1 - \frac{2M}{r}$

Start from:

$$r_* = r + 2M \ln\left(\frac{r}{2M} - 1\right) = \frac{1}{2}(v - u)$$

$$\frac{r_*}{2M} = \frac{r}{2M} + \ln\left(\frac{r}{2M} - 1\right) = \frac{1}{4M}(v - u)$$

$$\boxed{e^{\frac{r_*}{2M}}} = \underbrace{e^{\frac{r}{2M} + \ln\left(\frac{r}{2M} - 1\right)}}_{} = e^{(v-u)/4M}$$

$$e^{\frac{r}{2M}} e^{\ln\left(\frac{r}{2M} - 1\right)} = e^{r/2M} \underbrace{\left(\frac{r}{2M} - 1\right)}_{\frac{r}{2M}\left(1 - \frac{2M}{r}\right) = \frac{r}{2M} f} = \boxed{e^{r/2M} \frac{r}{2M} f}$$

so that:

$$f = \frac{2M}{r} e^{-r/2M} e^{(v-u)/4M}$$

and we have:

$$ds^2 = -\frac{2M}{r} e^{-r/2M} e^{(v-u)/4M}\, du\, dv$$

regular at $r = 2M$ (but not at $r = 0 \ldots$)

$ds^2 = -e^{v-u}\,du\,dv$

2) Now, motivated by the Rindler change to eliminate the e^{v-u} term we define

Schwarzschild:

$$ds^2 = -\frac{2M}{r}e^{-r/2M}e^{(v-u)/4M}$$

$$U = -e^{-u/4M}, \quad -\infty < U < 0$$

$$V = e^{v/4M} \quad 0 < V < \infty$$

$$r = 2M, \quad u \to \infty, \quad v \to -\infty$$

Rindler

$$ds^2 = -e^{v-u}\,du\,dv$$

$$U = -e^{-u}$$

$$V = e^{u}$$

$$U = 0, V = 0$$

$$dU = \frac{1}{4M}e^{-u/4M}\,du$$

$$dV = \frac{1}{4M}e^{v/4M}\,dv$$

$$dU\,dV = \frac{e^{(v-u)/4M}}{16M^2}\,du\,dv \qquad 16M^2\,dU\,dV = e^{(v-u)/4M}\,du\,dv$$

so that:

$$\boxed{ds^2 = -\frac{32M^3}{r}e^{-r/2M}\,dU\,dV} \quad \begin{array}{l}-\infty < U < 0\\ 0 < V < \infty\end{array}$$

No singularity at $r = 2M \Leftrightarrow U = V = 0$

3) Extension: So we can extend:

$$ds^2 = -\frac{32M^3}{r}e^{-r/2M}\,dU\,dV \qquad \begin{array}{l}-\infty < U < \infty\\ -\infty < V < \infty\end{array}$$

Define T, X coordinates:

$$T = \frac{U+V}{2} \qquad X = \frac{V-U}{2}$$

Then we have:

$$ds^2 = \frac{32M^3 e^{-r/2M}}{r}\left(-dT^2 + dX^2\right) + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2),$$

Inverse change of coordinates:

$$\frac{-u}{4M} = \ln(-U)$$

$$\begin{cases} t = \frac{1}{2}(u+v); \\ r_* = \frac{1}{2}(v-u) \end{cases} \quad \begin{cases} -e^{-u/4M} = U \\ e^{v/4M} = V \end{cases} \quad \begin{cases} U = T - X \\ V = T + X \end{cases}$$

$$\frac{v}{4M} = \ln V$$

- $$(-VU) \overset{?}{=} -(T+X)(T-X) = X^2 - T^2$$

$$-e^{-u/4M} e^{v/4M} = -e^{(v-u)/4M}$$

$$\frac{v-u}{4M} = \frac{r_*}{2M} = \frac{r}{2M} + \ln\left(\frac{r}{2M} - 1\right)$$

$$e^{\frac{v-u}{4M}} = e^{r/2M}\left(\frac{r}{2M} - 1\right)$$

$$e^{r/2M}\left(\frac{r}{2M} - 1\right) = X^2 - T^2$$

- $$t = \frac{1}{2}(u+v) = \frac{1}{2}\left(-4M \ln(-U) + 4M \ln V\right) = 2M\left(\ln\left(\frac{-1}{U}\right) + \ln V\right) =$$

$$= 2M \ln\left(\frac{-V}{U}\right) = 2M \ln\left(\frac{T+X}{-(T-X)}\right) = 2M \ln\left(\frac{X+T}{X-T}\right) =$$

$$= 2M \ln\left(\frac{1 + T/X}{1 - T/X}\right) \underset{\uparrow}{=} 4M \tanh^{-1}\left(\frac{T}{X}\right)$$

$$\tanh^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

$$\frac{t}{2M} = 2\tanh^{-1}\left(\frac{T}{X}\right)$$

- That is:

$$\boxed{\begin{aligned} e^{r/2M}\left(\frac{r}{2M} - 1\right) &= X^2 - T^2 \\ \frac{t}{2M} &= 2\tanh^{-1}\left(\frac{T}{X}\right) \end{aligned}}$$

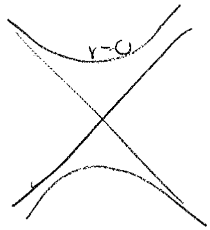
- $r =$ cte $\longrightarrow$ observers at fixed r

  $X^2 - T^2 =$ cte $\longrightarrow$ hyperboloids

- $t =$ cte $\longrightarrow$ slicing

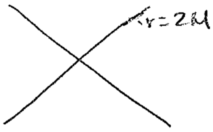
$$\alpha = \tanh\left(\frac{t}{2M}\right)$$

$$\frac{T}{X} = \text{cte} \qquad T = \alpha X$$



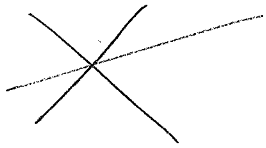
$r=0 \qquad X^2 - T^2 = -1$

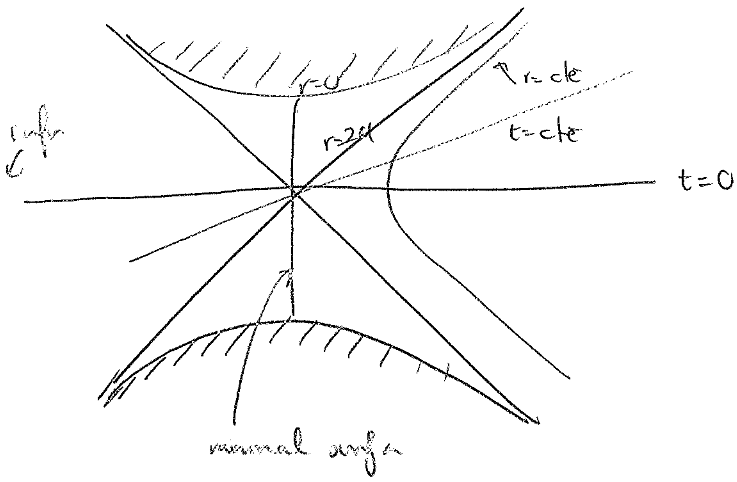
$r = 2M \qquad X^2 = T^2 \;;\; X = \pm T$



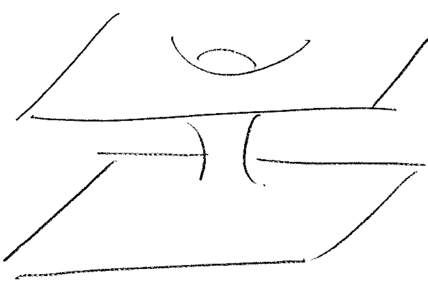
$t=0 \qquad T=0$

$t =$ cte





$t=0$



Einstein-Rosen bridge

Minkowski conformal compactification

- Minkowski metric

$$ds^2 = -dt^2 + dr^2 + r^2 \underbrace{d\Omega^2}_{d\theta^2 + \operatorname{sen}^2\theta\, d\varphi^2} \; ; \; -\infty < t < \infty, \; 0 < r < \infty$$

- Change to null coordinates

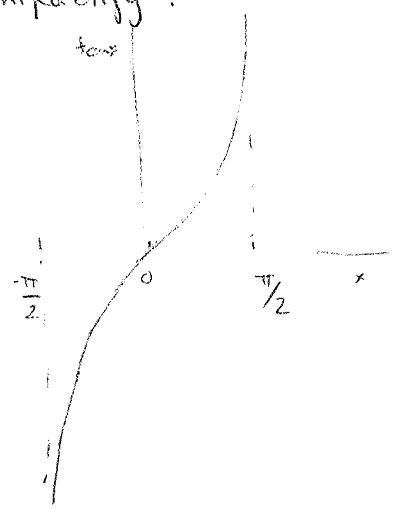
$$\begin{cases} u = t - r \\ v = t + r \end{cases} \qquad t = \frac{1}{2}(u+v) \qquad r = \frac{1}{2}(v-u) \qquad dt = \frac{1}{2}(du+dv) \qquad dr = \frac{1}{2}(dv-du)$$

$$ds^2 = -\frac{1}{4}\left(\cancel{du^2} + \cancel{dv^2} + 2du\,dv\right) + \frac{1}{4}\left(\cancel{dv^2} - 2dv\,du + \cancel{du^2}\right) +$$

$$+ \frac{1}{4}(v-u)^2 d\Omega^2 =$$

$$= -du\,dv + \frac{1}{4}(v-u)^2 d\Omega^2 \; ; \; -\infty < u < \infty$$

- Compactify :



$$u = \tan U \qquad -\pi/2 < U < \pi/2$$
$$v = \tan V \qquad -\pi/2 < V < \pi/2$$

$$u = \frac{\operatorname{sen} U}{\cos U} \qquad du = \frac{\overbrace{\cos^2 U + \operatorname{sen}^2 U}^{1}}{\cos^2 U}\, dU$$

$$dv = \dots = \frac{dV}{\cos^2 V}$$

$$v - u = \tan V - \tan U = \frac{\operatorname{sen}(V-U)}{\cos U \cos V}$$

$$\left(\frac{\operatorname{sen}(V-U)}{\cos U \cos V} = \frac{\operatorname{sen} V \cos U - \cos V \operatorname{sen} U}{\cos U \cos V} = \frac{\operatorname{sen} V}{\cos V} - \frac{\operatorname{sen} U}{\cos U} = \tan V - \tan U \right)$$

$$ds^2 = \frac{-1}{\cos^2 U \cos^2 V}\, dU\,dV + \frac{1}{4}\frac{\operatorname{sen}^2(V-U)}{\cos^2 U \cos^2 V}\, d\Omega^2 =$$

$$= \frac{1}{4\cos^2 U \cos^2 V}\left(-4\,dU\,dV + \underbrace{\operatorname{sen}^2(V-U)}_{V-U>0}\, d\Omega^2\right) \qquad -\frac{\pi}{2} < V < \pi/2 \; , \; -\frac{\pi}{2} < U < \pi/2$$

Define $\Omega \equiv 2\cos U \cos V$

$$d\tilde{s}^2 = -4\,dU\,dV + \operatorname{sen}^2(U-V)\,d\Omega^2 \qquad -\pi/2 < V < \pi/2\,, \quad -\pi/2 < U < \pi/2$$

$$ds^2 = \Omega^{-2}\,d\tilde{s}^2$$

$\Omega = 0$ at $U = \pm\frac{\pi}{2}$, $V = \pi/2$

- Change now to:

$-\pi < \psi < \pi$

$-\pi < T < \pi$

but if $\psi = 0$

$\operatorname{sen}^2(V-U) = 0$

and the metric degenerates at an inner point

↓

$0 < \psi < \pi$

$$\psi = V - U \qquad V = \frac{1}{2}(\psi + T) \qquad dV = \frac{1}{2}(d\psi + dT)$$

$$T = V + U \qquad U = \frac{1}{2}(T - \psi) \qquad dU = \frac{1}{2}(dT - d\psi)$$

$$\cos T + \cos\psi = \cos(V+U) + \cos(V-U) = \cos V\cos U - \operatorname{sen}V \operatorname{sen}U +$$
$$+ \cos V \cos U + \operatorname{sen}V\operatorname{sen}U =$$
$$= 2\cos V \cos U = \boxed{\Omega}$$

$$d\tilde{s}^2 = -dT^2 + \underbrace{d\psi^2 + \operatorname{sen}^2\psi\, d\Omega^2}_{\text{metric on } S^3} \longrightarrow \text{Einstein cilinder } \mathbb{R}\times S^3$$

$$w^2 + x^2 + y^2 + z^2 = 1 \qquad\qquad 0 < T < \pi$$

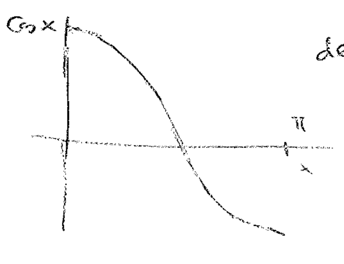
$$\begin{cases} w = \cos\psi & 0 \le \psi < \pi \\ x = \operatorname{sen}\psi\cos\theta & 0 \le \theta < \pi \\ y = \operatorname{sen}\psi \operatorname{sen}\theta\cos\varphi & 0 \le \varphi < 2\pi \\ z = \operatorname{sen}\psi\operatorname{sen}\theta\operatorname{sen}\varphi \end{cases}$$

$$d\sigma^2 = dw^2 + dx^2 + dy^2 + dz^2 \overset{S^3}{=} d\psi^2 + \operatorname{sen}^2\psi\, d\Omega^2$$

- Change from (t, r) to (T, ψ)

$$t = \frac{\operatorname{sen} T}{\underbrace{\cos T + \cos\psi}_{\Omega}} \qquad r = \frac{\operatorname{sen}\psi}{\underbrace{\cos T + \cos\psi}_{\Omega}} \qquad \text{sen}$$

$\Omega > 0$ in: $0 < \psi < \pi$;



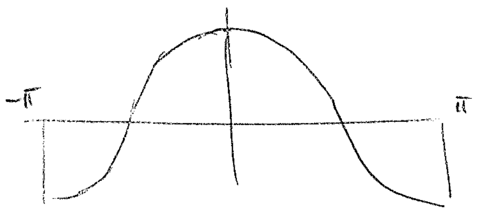
decreasing $\quad y > x \Rightarrow \cos y < \cos x$

We need to impose $\Omega > 0$

$$\cos T + \cos \chi > 0$$

$$\cos T > -\cos \chi$$

We constrain $0 < \chi < \pi$ and note that $\cos x$ in mon. increasing in $-\pi < x < 0$ and mon. decreasing in $0 < x < \pi$ so that



$T > \pi - \chi > 0$

$\cos T < \cos(\pi - \chi) = -\cos \chi$ wrong

$\underline{T < \pi - \chi} > 0$

$\cos T > \cos(\pi - \chi) = -\cos \chi$ ok

$\underline{T > \chi - \pi} < 0$

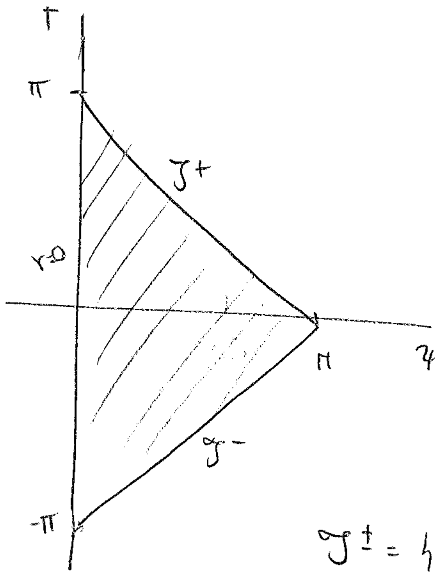
$\cos T > \cos(\chi - \pi) = -\cos \chi$ ok

$T < \chi - \pi < 0$

$\cos T < \cos(\chi - \pi) = -\cos \chi$ wrn

$$\boxed{\begin{array}{c} \chi - \pi < T < \pi - \chi \\ 0 < \chi < \pi \end{array}}$$

$r \sim \sin \chi$



$\mathcal{J}^{\pm} = \{0 < \chi < \pi,\ T = \pm(\pi - \chi)\} \longrightarrow$ endpoint of null geodesics

$i^0 = \{T = 0,\ \chi = \pi\} \longrightarrow$ " " spacelike "

$i^{\pm} = \{T = \pm\pi,\ \chi = 0\} \longrightarrow$ " " timelike "

⋮

Conformal compactification of Schwarzschild

We remind:

$$ds^2 = -\frac{32M^3}{r} e^{-r/2M} dU\, dV + r^2 d\Omega^2$$

and notice

$$UV = -e^{(v-u)/4M} = -e^{\frac{r^*}{2M}} = -e^{r/2M + \ln\left(\frac{r}{2m}-1\right)} = -\left(\frac{r}{2m}-1\right)e^{r/2M} =$$

$$= \frac{-1}{2m}(r-2m)e^{r/2m}$$

As in Minkowski, introduce

$$U = \tan \bar{U} \qquad -\frac{\pi}{2} < \bar{U} < \pi/2$$

$$V = \tan \bar{V} \qquad -\frac{\pi}{2} < \bar{V} < \pi/2$$

$$dU = \frac{1}{\cos^2 \bar{U}} d\bar{U} \qquad dV = \frac{1}{\cos^2 \bar{V}} d\bar{V}$$

so that:

$$ds^2 = -\frac{32M^3}{r} e^{-r/2M} \frac{1}{\cos^2\bar{U}\cos^2\bar{V}} d\bar{U}\, d\bar{V} + r^2 d\Omega =$$

$$= \frac{1}{\cos^2\bar{U}\cos^2\bar{V}}\left(-\frac{32M^3}{r} e^{-r/2M} d\bar{U} d\bar{V} + r^2 \cos^2\bar{U}\cos^2\bar{V}\, d^2\Omega\right)$$

Again define $\Omega = \cos\bar{U}\cos\bar{V}$, so

$$\boxed{\begin{aligned} ds^2 &= \Omega^{-2} d\tilde{s}^2 \\ d\tilde{s}^2 &= -\frac{32M^3}{r} e^{-r/2M} d\bar{U} d\bar{V} + r^2 \cos^2\bar{U}\cos^2\bar{V}\, d^2\Omega \end{aligned}}$$

for $r > 0$

Note that here $-\pi/2 < \bar{U} < \pi/2$, $-\frac{\pi}{2} < \bar{V} < \pi/2$ makes the factor of in front of $d^2\Omega$ non-vanishing. So, as a difference with Minkowski, there is no restriction on $V-U$

Introduce then:

$$\begin{cases} T = \bar{V} + \bar{U} \\ \chi = \bar{V} - \bar{U} \end{cases}$$

$T \in (-\infty, \infty)$

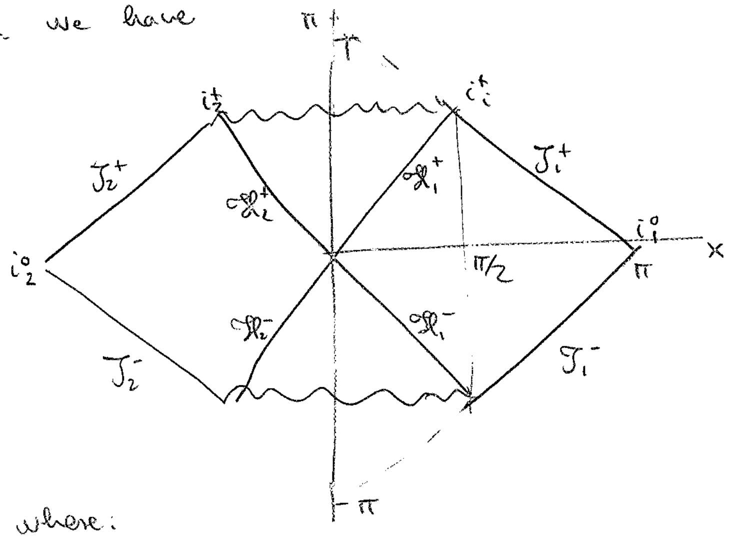
$$\begin{cases} \bar{V} = \frac{1}{2}(T + \chi) \\ \bar{U} = \frac{1}{2}(\bar{U} - \chi) \end{cases}$$

$\chi = (-\pi, \pi) \longrightarrow$ two copies (no restriction on $V-U$!)

as in Minkowski

$0 < r < \infty$

Then we have



where:

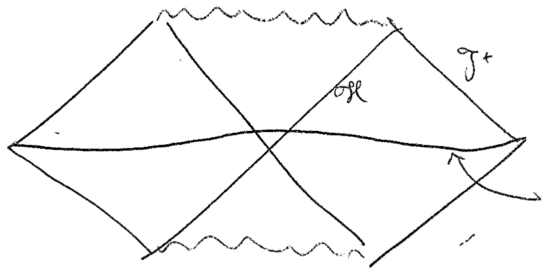
$$\begin{cases} \mathcal{J}_1^+ = \{ \bar{V} = \frac{1}{2}\pi \,;\; T = -\chi \,, \quad 0 < \chi < \pi \} \\ \mathcal{J}_1^- = \{ \bar{U} = -\frac{1}{2}\pi \} \\ \mathcal{J}_2^+ = \{ \bar{U} = \frac{1}{2}\pi \\ \mathcal{J}_2^- = \{ \bar{V} = -\frac{1}{2}\pi \} \end{cases}$$

$i_1^0 = \{ T = 0, \chi = \pi \}$; $i_2^0 = \{ T = 0, \chi = -\pi \}$

$i_1^\pm = \{ T = \pm\pi \,,\; \chi = \pi/2 \}$

$i_2^\pm = \{ T = \pm\pi \,,\; \chi = -\pi/2 \}$

- Review Schwarzschild compactification as a model



Cauchy surface → Globally hyperbolic

- Conformal compactification picture
  - $(M, g_{\mu\nu})$ physical
  - $(\tilde{M}, \tilde{g}_{\mu\nu})$ $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$
  - Infinity = boundary of $\partial\tilde{M}$ characterized by $\Omega = 0$.

$$\partial\tilde{M} = \mathscr{I}^{\pm} \cup i^0 \cup i^{\pm}$$

Asymptotic simplicity:

$(M, g_{\mu\nu})$ is asymptotically simple if there exists $(\tilde{M}, \tilde{g}_{\mu\nu})$

i) M is an open submanifold of $\tilde{M}$ with smooth boundary $\partial\tilde{M}$.

ii) There exists a smooth scalar Ω on $\tilde{M}$ such that.

$$\Omega > 0, \ \tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$$
and $\Omega = 0$ & $\partial_\mu \Omega \neq 0$ on $\partial\tilde{M}$

iii) Every null geodesic in M begins and ends in $\partial\tilde{M}$.

An asymptotically simple spacetime is asymptotically flat if, in addition, Einstein equations are satisfied.

- Black Hole region and Event Horizon.

  - BH region:

    Def 1: The BH region $\mathcal{B}$ is the complement in $\mathcal{M}$ of the causal past of future null infinity:

$$\mathcal{B} = \mathcal{M} - J^-(\mathscr{I}^+)$$

  - Event horizon:

    Def 2: The event horizon is the boundary of the BH region $\mathcal{B}$ in $\mathcal{M}$.

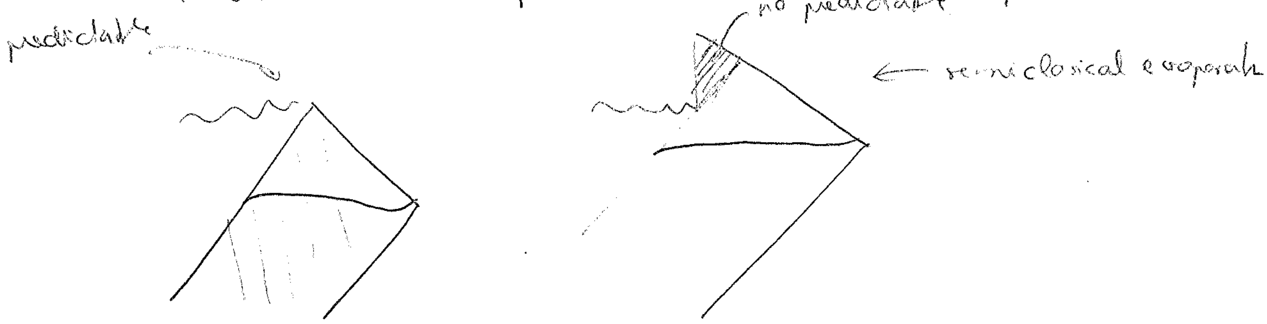
$$\mathcal{E} = \partial J^-(\mathscr{I}^+) \cap \mathcal{M}$$

i) Strongly asymptotically predictable spacetimes.

Def: A spacetime $(\mathcal{M}, g_{\mu\nu})$ is said to be strongly asymptotically predictable if, given the conformally rescaled spacetime $(\tilde{\mathcal{M}}, \tilde{g}_{\mu\nu})$ there is an open region $\tilde{V} \subset \tilde{\mathcal{M}}$ containing the closure of the boundary and exterior of the (would-be) BH region (i.e: $\overline{\mathcal{M} \cap J^-(\mathscr{I}^+)} \subset \tilde{V}$) that is globally hyperbolic ($\Leftrightarrow$ it admits a Cauchy hypersurface)

Then $(\mathcal{M} \cap \tilde{V}, g_{\mu\nu})$ is globally hyperbolic.

That is: we can predict in



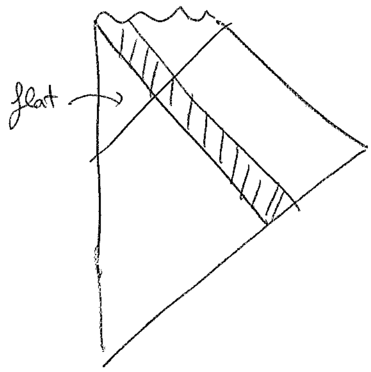
- Properties $\mathcal{E}$:

  i) Null hypersurface that does not "bifurcate". $\mathcal{E}$ is a null hypersurface in $\mathcal{M}$

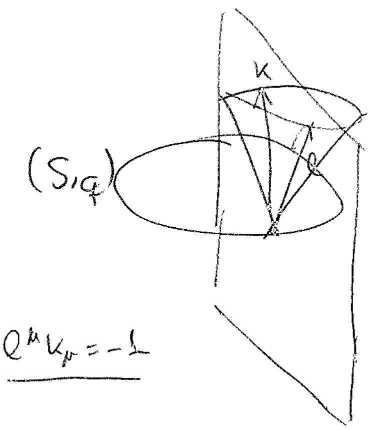
  ii) "Area law": the area of sections of $\mathcal{H}$ cannot decrease.

  iii) Global teleological properties of $\mathcal{E}$.

  iv) It enters the flat region: Vaydia spacetime



- Trapped region approach



$\ell^\mu k_\mu = -1$

- Spacelike closed (compact without boundary) surface

$$q_{\mu\nu} = g_{\mu\nu} + \ell_\mu k_\nu + k_\mu \ell_\nu$$

Volume element $dA = \sqrt{q}\, dx^1 \wedge dx^2$

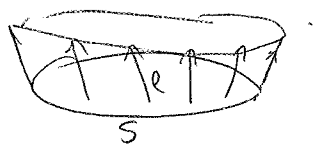
$$A = \int \sqrt{q}\, d^2x$$

$$q_{\mu\nu} \ell^\mu = 0$$
$$q_{\mu\nu} k^\mu = 0$$

Now we emit light from S in the outgoing and ingoing directions.

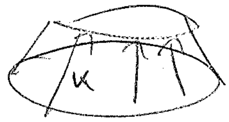
Consider the area of the frontlines:

$\mathcal{L}_\ell \epsilon_{\mu\nu}$ is a 2-form $\Rightarrow$ proportional to $\epsilon_{\mu\nu}$



$$\mathcal{L}_\ell \epsilon_{\mu\nu} \equiv \theta^{(\ell)} \epsilon_{\mu\nu} \Rightarrow$$

$$\Rightarrow \mathcal{L}_\ell \sqrt{q} = \theta^{(\ell)} \sqrt{q}$$



$$\mathcal{L}_\ell \sqrt{q} = ?$$

$$\delta \ln q = q^{\mu\nu} \delta q_{\mu\nu}$$

$$A = \int dA = \int \epsilon$$

$$\delta \ln\sqrt{q} = \frac{1}{2} q^{\mu\nu} \delta q_{\mu\nu} \qquad \delta\sqrt{q} = \frac{1}{2} q^{\mu\nu}\delta q_{\mu\nu}\sqrt{q}$$

$$\delta\ln\sqrt{q} = \frac{\delta\sqrt{q}}{\sqrt{q}}$$

$$\boxed{\mathcal{L}_\ell A = \int \mathcal{L}_\ell \epsilon = \int \mathcal{L}_\ell \sqrt{q} = \int \theta^{(\ell)} \sqrt{q}}$$

$$\delta \to \mathcal{L}_\ell$$

$$\mathcal{L}_\ell \sqrt{q} = \frac{1}{2} q^{\mu\nu} \underbrace{\mathcal{L}_\ell q_{\mu\nu}} \sqrt{q} =$$

$$\left(\begin{aligned} & q^{\mu\nu} \mathcal{L}_\ell (g_{\mu\nu} + \cancel{\ell_\mu k_\nu} + \cancel{k_\mu \ell_\nu}) = \\ & = q^{\mu\nu} \mathcal{L}_\ell g_{\mu\nu} = q^{\mu\nu} (\nabla_\mu \ell_\nu + \nabla_\nu \ell_\mu) \end{aligned} \right)$$

$$= \frac{1}{2} q^{\mu\nu} (\nabla_\mu \ell_\nu + \nabla_\nu \ell_\mu) \sqrt{q} = \underbrace{q^{\mu\nu} \nabla_\mu \ell_\nu}_{\theta^{(\ell)}} \sqrt{q}$$

$$\theta^{(\ell)} = \frac{1}{2} \frac{1}{\sqrt{q}} \mathcal{L}_\ell \sqrt{q} = q^{\mu\nu} \nabla_\mu \ell_\nu$$

Analogously:

$$\theta^{(k)} = \frac{1}{2} \frac{1}{\sqrt{q}} \mathcal{L}_k \sqrt{q} = q^{\mu\nu} \nabla_\mu \ell_\nu$$

Trapped surfaces: light cannot scape.

BH: $\theta^{(\ell)}<0$ $\theta^{(k)}<0$ Penrose

Outer trapped surface

$\theta^{(\ell)}<0$ Hawking

ℓ^μ outgoing

Marginally trapped Surfac

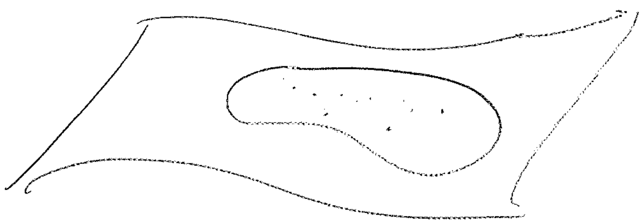
$\theta^{(\ell)}=0$; $\theta^{(k)}$

Marginally Outer Trapped surface (MOTS)

$\boxed{\theta^{(\ell)}=0}$

- Trapped region in a 3-slice Σ

$T: \{p \in \Sigma \,/\, p \in S,\ S \text{ trapped}\}$



Apparent Horizon: outermost bondary of the trapped region T.

Characterization (Theorem)

The Apparent Horizon is a MOTS $\theta^{(\ell)}=0$

Remark: The extension of this ideas beyond a slice Σ to a spacetime perspective is complicated and leads to unexpected results.

- Results relating the Event Horizon and (marginally) trapped surfaces.

Trapped surfaces are a footprint of the BH region $\mathcal{B}$:

Th. Let us assume a strongly predictable spacetime satisfying the weak and strong energy conditions $R_{ab}\ell^a\ell^b$.

i) Every TS S is contained in the BH region: $S \subset \mathcal{B}$

ii) Every MTS " " " " " " : $S \subset \mathcal{B}$

iii) Let Σ be an asymptotically flat Cauchy slice for the hyperbolic region $\mathcal{V}$. Let T be a trapped region in Σ i.e. $T \subset T_\Sigma \subset \Sigma$. Then T is contained in the BH region $T \subset \mathcal{B}\cap\Sigma$.

- Geometry of spacelike surfaces $S \subset \mathcal{M}$

- Deformation tensor: ℓ^μ, k^μ normal to S

$$\boxed{\Theta^{(\ell)}_{\mu\nu} = \frac{1}{2} \mathcal{L}_\ell q_{\mu\nu}} \qquad \Theta^{(k)}_{\mu\nu} = \frac{1}{2} \mathcal{L}_k q_{\mu\nu}$$

$$= q^\alpha_\mu q^\beta_\nu D_\alpha \ell_\nu$$

Symmetric: using ℓ^μ and k^μ normal to S (Frobenius th.)

- Expansion and shear

Trace $\Theta^{(\ell)} = q^{\mu\nu} \Theta^{(\ell)}_{\mu\nu}$; Traceless part $\sigma^{(\ell)}_{\mu\nu} = \Theta^{(\ell)}_{\mu\nu} - \frac{1}{2} \Theta^{(\ell)} q_{\mu\nu}$

$\Theta^{(k)} = q^{\mu\nu} \Theta^{(k)}_{\mu\nu}$; $\sigma^{(k)}_{\mu\nu} = \Theta^{(k)}_{\mu\nu} - \frac{1}{2} \Theta^{(k)} q_{\mu\nu}$

- Rotation 1-form

$$\Omega_\mu = - k^\nu q^\rho_\mu D_\rho \ell_\nu$$

Given an axial Killing vector ϕ^μ; $\boxed{J_{\mathrm{Kom}} = \frac{1}{8\pi} \int D_\mu \phi_\nu \, dS^{\mu\nu}}$

$$\boxed{J = \frac{1}{8\pi} \int \Omega_\mu \phi^\mu dA} \qquad \boxed{dS^{\mu\nu} = \frac{1}{2} (k^\mu \ell^\nu - \ell^\mu k^\nu) dA}$$

- Transformation properties

$\ell^\mu \to \ell'^\mu = f \ell^\mu$; $k^\mu \to k'^\mu = f^{-1} k^\mu$ $\qquad \ell'^\mu k'_\mu = -1$

$$\Theta^{(\ell')} = f \Theta^{(\ell)} \qquad \sigma^{(\ell')}_{\mu\nu} = f \sigma^{(\ell)}_{\mu\nu}$$

$$\Theta^{(k')} = f^{-1} \Theta^{(k)} \qquad \sigma^{(k')}_{\mu\nu} = f^{-1} \sigma^{(k)}_{\mu\nu}$$

$$\Omega^{(\ell')}_\mu = \Omega^{(\ell)}_\mu + D_\mu \ln f$$

$$J' = \frac{1}{8\pi} \int \Omega^{(\ell')}_\mu \phi^\mu dA = \frac{1}{8\pi} \int \left(\Omega^{(\ell)}_\mu \phi^\mu + D_\mu (\ln f) \phi^\mu \right) dA =$$

$$= \frac{1}{8\pi} \int \Omega^{(\ell)}_\mu \phi^\mu dA - \frac{1}{8\pi} \int \ln f \underbrace{D_\mu \phi^\mu}_{0} = J$$

$$D_\mu \phi_\nu + D_\nu \phi_\mu = 0 \Longrightarrow D_\mu \phi^\mu = 0$$

Raychaudhuri equation:

$$\mathcal{L}_\ell \Theta^{(\ell)} = \kappa^{(\ell)}\Theta^{(\ell)} - \frac{1}{2}\Theta^{(\ell)2} - \sigma^{(\ell)}_{\mu\nu}\sigma^{\mu\nu(\ell)} - 8\pi T(\ell,\ell)$$

- Energy conditions
  - Weak energy condition: $T_{\mu\nu}\xi^\mu\xi^\nu \geq 0$ $\quad \xi^\mu$ timelike
  - " null " " : $T_{\mu\nu}\ell^\mu\ell^\nu \geq 0$ $\quad \ell^\mu$ null

    } energy of density is non negative
  - Strong energy condition $T_{\mu\nu}\xi^\mu\xi^\nu \geq -\frac{1}{2}T$
  - Dominant energy condition: $-T^\mu{}_\nu\xi^\nu$ causal (timelike or null)

    (null) $\underbrace{-T^\mu{}_\nu\ell^\nu}_{\text{energy momentum 4-current}}$ " " " "

    } speed of energy flow always less than light like

Dynamics: What is the fate of dynamical gravitational collapse (as opposed to "eternal BHs" such as Schwarzschild or Kerr)

1) Singularity theorems

- Singularity.

  Defining what a singularity is is a very complicate endeavour.
  Not only diverging curvature invariants involving $R^\rho{}_{\sigma\mu\nu}$ and its covariant derivatives are relevant. There are other types of singularities including vanishing curvature, e.g. conical singularities.
  The idea is that when we remove these 'points' from the spacetime, there should remain a hole that cannot be removed by immersing our spacetime isometrically in a larger regular spacetime.

  We characterise a spacetime as singular if it is geodesically incomplete, that is, if there are geodesics of finite affine length that cannot be further extended.

- Singularity theorems.

They involve Einstein equations, some energy condition and an additional feature. We give here Penrose singularity theorem in gravitational collapse involving trapped surfaces.

→ They show that singularities are generic objects, not relying on fine-tuned symmetry requirements (as in Newtonian theory)

Th: (Penrose '65)

Let $(M, g_{\mu\nu})$ be a connected, globally hyperbolic spacetime with a non-compact Cauchy surface Σ. Suppose $R_{\mu\nu} k^\mu k^\nu \geq 0$ for all null k^μ (e.g. Einstein + weak/strong energy conditions). Suppose that M contains a (strictly) trapped surface S. Let $\theta_0 < 0$ be the maximum value of $\theta^{(\ell)}$ and $\theta^{(n)}$! Then at least one inextendible future directed orthogonal null geodesic from S has affine length no greater than $\frac{2}{|\theta_0|}$.

2) Cosmic censorship

We want to have a predictive theory: the singularity should not be seen by (at least) distant observers. The result of a gravitational collapse should be an asymptotically predictable spacetime.

Weak Cosmic Censorship conjecture (physical formulation)

The complete gravitational collapse of a body always results in a BH rather than in a naked singularity, i.e. all singularities are hidden within the BH region.

WCCC (math.) Let $(\Sigma, \gamma_{ab}, K_{ab})$ be an asymptotically flat initial data with (Σ, γ_{ab}) a complete Riemannian manifold; let the matter sources be such that $T_{\mu\nu}$ satisfies the dominant energy condition. Assume appropriate fall-off conditions for the matter at infinity. Then the maximal Cauchy development of this initial data is an asymptotically flat, strongly asymptotically predictable spacetime

Remark 1: The collapse of an scalar field can be fine-tuned to produce a null+naked singularity. This was discovered in the attempt to produce arbitrarily small BHs and led to the discover of critical phenomena in GR (Choptuik, Christodoulou).
The initial data are however non-generic: the property is lost for infinitesimal variations of the data.
Weak Cosmic Censorship is reformulated for generic initial data.

Remark 2: Strong Cosmic Censorship. About Cauchy horizon → wait Kerr

- Status now
  1) Singularity theorems
  2) Cosmic Censorship

3) Stability of BH spacetimes → Once the BH is formed one eventually reaches a stationary state.
Conjecture. Active Research.

4) Uniqueness theorems. let us wait to Kerr.

Then we will derive Penrose conjecture for the relation of the BH area and the ADM mass in a BH spacetime.

Dynamical quasilocal modelling of BH horizons

- <u>Trapping horizon $\mathcal{H}$:</u> worldtube of MOTS:

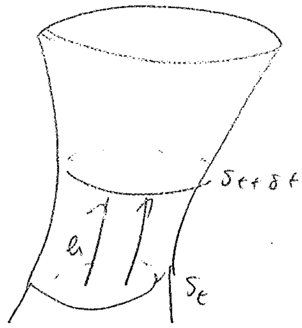
$$\mathcal{H} = \bigcup_t S_t \qquad S_t \text{ MOTS i.e. } \theta^{(\ell)} = 0$$

Two geometric conditions

- Singularity in the future $\theta^{(k)} < 0$ (future)
- Outer boundary of the "trapped region"

$$\delta_k \theta^{(\ell)} < 0$$

We can define the evolution vector on the $\mathcal{H}$, transporting a MOTS S_t into a MOTS $S_{t+\delta t}$, normal to S_t and tangent to $\mathcal{H}$



$$h = \ell^\mu - C k^\mu$$

$h_\mu h^\mu = 2C$ $\quad$ $C > 0$ $\mathcal{H}$ spacelike } we better are in this case

$C = 0$ $\mathcal{H}$ null

$C < 0$ $\mathcal{H}$ timelike

Let us make explicit the trapping horizon condition:

$$\mathcal{L}_h \theta^{(\ell)} = 0$$

$$\mathcal{L}_\ell \theta^{(\ell)} - \mathcal{L}_{Ck} \theta^{(\ell)} = 0$$

Let us assume spherical symmetry $C \neq C(\theta, \varphi)$

Then:

$$C = \frac{\mathcal{L}_\ell \theta^{(\ell)}}{\mathcal{L}_k \theta^{(\ell)}} \overset{\text{Raychaudhuri}}{=} - \frac{|\sigma^{(\ell)}|^2 + T_{\mu\nu}(\ell^\mu, \ell^\nu)}{\mathcal{L}_k \theta^{(\ell)}}$$

Then weak energy condition $T_{\mu\nu} \ell^\mu \ell^\nu \geq 0$ + outer condition ...

$\longrightarrow \quad C \geq 0 \longrightarrow \begin{cases} C = 0 \Leftrightarrow \text{null: stationary} \to h^\mu = \ell^\mu \to \underline{\text{Raych.}} \Rightarrow \sigma_{\mu\nu} = 0 \\ C > 0 \Rightarrow \text{spacelike: } \underline{\text{dynamical}} \end{cases}$

$\mathcal{L}_\ell q_{\mu\nu} = 0$

$T_{\mu\nu}(\ell^\mu \ell^\nu) = 0$

Metric type of $\mathcal{H}$ and dynamical states

<u>Area growth</u>

$$A = \int dA = \int \epsilon$$

$$\dot{A} = \mathcal{L}_h A = \int \underset{\substack{\| \\ \theta^{(\ell)} - C\theta^{(n)}}}{\theta^{(h)}} \epsilon = \int (\cancel{\theta^{(\ell)}}^{\to 0} - C\theta^{(n)})\, dA = -C \overbrace{\int \underset{\substack{\uparrow \\ \text{future}}}{\theta^{(n)}}\, dA}^{<0}$$

$\dot{A} = 0$ for $C = 0$ $\to$ stationary

$\dot{A} > 0$ for $C > 0$ $\to$ dynamical

Null case $\to$ Isolated Horizon formalism

Spacelike case $\to$ Dynamical Horizons

Very interesting area of research.

- Line element of Kerr

$$ds^2 = -\left(\frac{\Delta - a^2 \sin^2\theta}{\Sigma}\right) dt^2 - \frac{2a \sin^2\theta \,(r^2+a^2-\Delta)}{\Sigma} dt\, d\varphi$$

$$+ \frac{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta}{\Sigma} \sin^2\theta\, d\varphi^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma\, d\theta^2$$

with $\Sigma = r^2 + a^2 \cos^2\theta$ $\qquad a = \frac{J}{M}$

$\Delta = r^2 + a^2 - 2Mr$

Killing vectors: $\xi^\mu = (\partial_t)^\mu$

$\psi^\mu = \left(\frac{\partial}{\partial\varphi}\right)^\mu$

Komar quantities $M = -\frac{1}{8\pi}\int \nabla_\mu \xi_\nu \, dS^{\mu\nu}$

$J = \frac{1}{8\pi}\int \nabla_\mu \psi_\nu \, dS^{\mu\nu}$

$a = \frac{J}{M}$

- Singularity at $\Sigma = 0$, $r^2 + a^2 \cos^2\theta = 0$ (ring singularity)

- No singularity at $\Delta = 0$ For $a \leq M$:

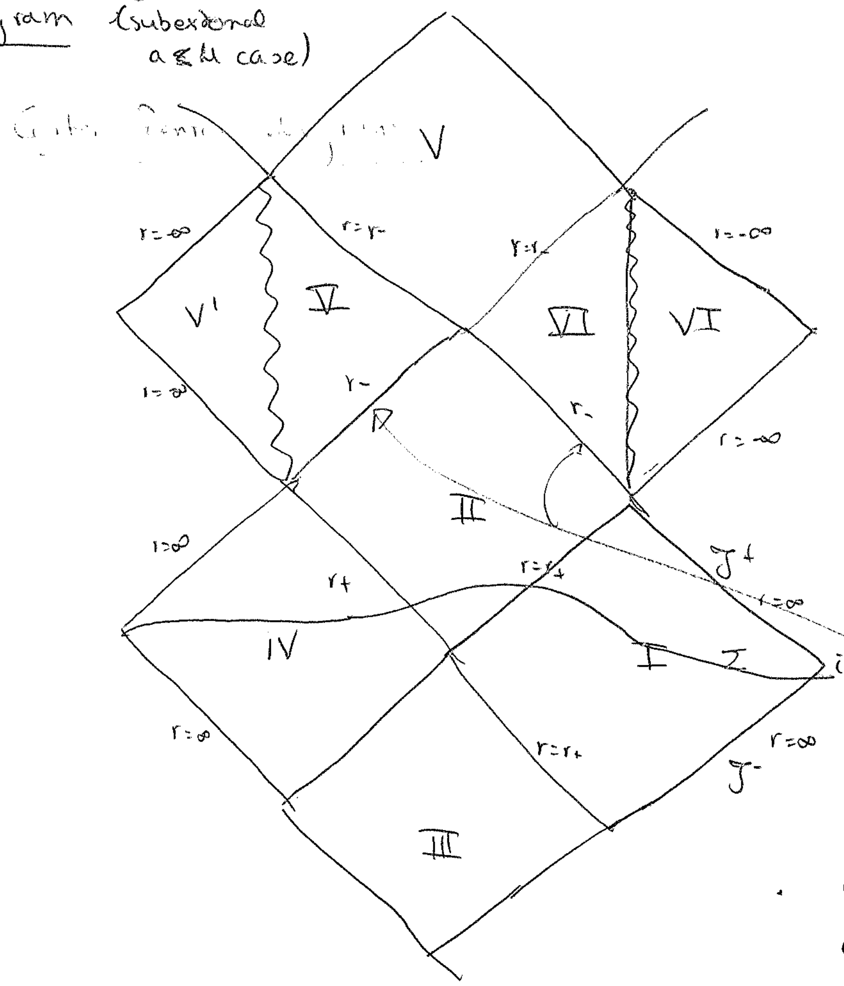
$r_+ = M + \sqrt{M^2 - a^2} \longrightarrow$ Event Horizon

$r_- = M - \sqrt{M^2 - a^2} \longrightarrow$ Cauchy Horizon

For the superextremal case $a > M$

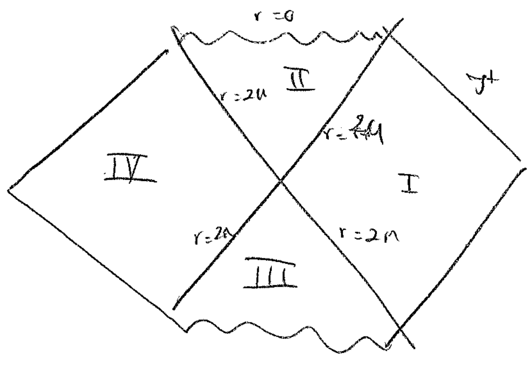
there is no BH region: naked singularity. According to Weak Cosmic Censorship, it cannot be final state of a total gravitational collapse.

Carter-Penrose diagram (subextremal a & M case)



- Cauchy Horizon
  Einstein equations do not determine the spacetime beyond that boundary. other

Compare with



- Strong Cosmic Censorship:
  Idea: Cauchy horizon is unstable and it becomes a true singularity in the generic case: blue mass effect (Any point in $r=r_-$ "sees" the whole Cauchy surface Σ).

- Kerr family
  - Two parameter family $M, J \Leftrightarrow M, a \Leftrightarrow A, J$
  - It can be extended to the charged case: Kerr-Newman
  - The non-rotating charged case, namely Reissner-Nordstrom shares qualitative features with Kerr in a spherically symmetric context. Birkhoff theorem can be generalized to this case

Relevant expressions

$$A = 8\pi\left(M^2 + \sqrt{M^4 - a^4}\right)$$

Note Schwarzschild:

$$A = 16\pi M^2$$

Define irreducible mass (see later):

$$M_{ir} = \sqrt{\frac{A}{16\pi}}$$

$$M = \sqrt{\frac{A}{16\pi} + \underbrace{\frac{4\pi J^2}{A}}_{\text{rotational energy}\dots}} \qquad M = M(A, J)$$

- killing vectors

$\xi = \partial_t$, axial vector $\phi^\mu = \partial_\varphi$,

Note that ξ^μ is not always timelike:

$$\xi^\mu \xi_\mu = g_{00} = \frac{a^2 \sin^2\theta - \Delta}{\Sigma} = r^2 \sin^2\theta - r^2 - a^2 + 2Mr = \frac{-r^2\cos^2\theta - a^2 + 2Mr}{\Sigma}$$

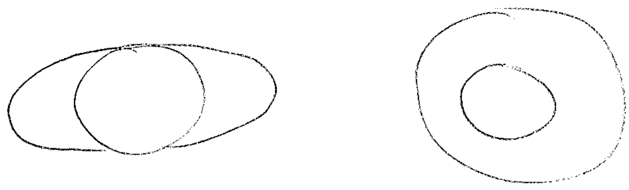
If $r^2\cos^2\theta + a^2 < 2Mr$, $\xi^\mu \xi_\mu$ spacelike

Def: Ergosphere: region in which the otherwise timelike killing vector becomes spacelike.

All observers in the ergosphere are forced to rotate. One can show

$u^\mu = \left(\frac{dt}{d\tau}, \dots, \frac{d\phi}{d\tau}\right)$

$\frac{d\phi}{d\tau} > 0 \to$ rotate



Defining an observer normal to the $t = \text{const}$ slice:

$$u^\mu = \frac{-\nabla^\mu t}{(\nabla_\nu t \nabla^\nu t)^{1/2}} \qquad u^\mu \sim n^\mu$$

Thus observers have vanishing angular momentum $L = u^\mu \phi_\mu = u_\mu \phi^\mu = \nabla_\mu t (\partial_\phi)^\mu = 0$

and rotate at:

$$\Omega = \frac{d\varphi}{dt} = \frac{d\tau}{dt}\frac{d\varphi}{d\tau} = \frac{u^\varphi}{u^t} = \underset{\text{do it!}}{\frac{g^{\varphi t}}{g^{tt}} = -} \frac{g_{t\varphi}}{g_{\varphi\varphi}} = \frac{a(r^2 + a^2 - \Delta)}{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta}$$

On the horizon:

$$\Omega_H = \frac{a}{r_+^2 + a^2}$$ Interpreted as an angular velocity of the horizon

In particular, the Killing vector:

$$\chi^\mu = \xi^\mu + \Omega_H \phi^\mu$$

is the Killing field becoming null at the horizon.

Killing horizon: Null surface generated by a (null) Killing vector that is timelike otherwise.

Uniqueness theorems.

Series of results that essentially state that the only stationary vacuum spacetime admitting a Killing horizon (satisfying appropriate regularity conditions) is subextremal Kerr.

No uniqueness result in the superextremal case.

This is the essence of NO-HAIR theorems: the general final result of ~~total~~ gravitational collapse (assuming that dynamics drive the system to stationarity) is parametrized by only two parameters.

A note on BH thermodynamics:

$M = M(A, J)$

$$\delta M = \frac{\delta M}{\delta A}\delta A + \frac{\delta M}{\delta J}\delta J$$

$$\frac{\delta M}{\delta A} = \frac{\kappa}{8\pi} = \frac{1}{4M} - \frac{16\pi^2 J^2}{MA^2} = \frac{A^2 - 64\pi^2 J^2}{4MA^2} = \frac{A^2 - (8\pi J)^2}{4MA^2}$$

$$\frac{\delta M}{\delta J} = \Omega = \frac{4\pi J}{MA} =$$

$$\boxed{\Omega_H = \frac{a}{r_+^2 + a^2}} = \frac{\frac{J}{M}}{M^2 + M^2 - \cancel{a^2} + 2\sqrt{M^2 - a^2} + \cancel{a^2}} = \frac{J}{M(2M^2 + 2\sqrt{M^2 - a^2})} =$$

$$\left(A = 8\pi(M^2 + \sqrt{M^2 - a^2})\right) = \frac{J}{2M(M^2 + \sqrt{M^2 - a^2})} = \frac{8\pi J}{2M\underbrace{8\pi(M^2 - \sqrt{M^2 - a^2})}_{A}}$$

$$= \boxed{\frac{4\pi J}{MA}}$$

Therefore:

$$\boxed{\delta M = \frac{\kappa}{8\pi}\delta A + \Omega_H \delta J}$$

Where: κ: $\underbrace{\chi^\mu \nabla_\mu \chi^\alpha = \kappa \chi^\alpha}$

and Ω_H: $\chi^\mu = \xi^\mu + \Omega_H \phi^\mu$

Note $\delta A \geq 0$ (Hawking area law)

It suggests $A \sim$ entropy. $\longrightarrow S_A$

For this, κ should be $\sim$ a temperature

"Confirmed" by Hawking radiation:

A quantum field on a BH background is unstable and radiate as a black body with $\boxed{T_H = \frac{\kappa}{2\pi}}$

Therefore $\boxed{S = \frac{A}{4}}$

Extraction of energy from a BH

Penrose process

Crucial element: ergosphere.

1) Consider a particle with 4-momentum $p^\mu = m u^\mu$
Then, the energy is given by the conserved quantity

$$E = -p^\mu \xi_\mu$$

needs not to be positive if ξ^μ is not timelike, namely in the ergosphere.

2) Consider a distant particle
with momentum p_0^μ. Then the conserved energy in free-fall is

$$E_0 = -p_0^\mu \xi_\mu .$$

3) Consider that once in the ergosphere the particle breaks in two pieces with momentum:

$$p_0^\mu = p_1^\mu + p_2^\mu \quad \text{(local conservation of momentum)}$$

Therefore:

$$E_0 = E_1 + E_2$$

The breaking can be arranged in such a way that

$$E_1 < 0 \text{ in the ergosphere.}$$

Then, the fragment E_1 falls into the BH, that reduces its energy and the second fragment escapes to infinity with $E_2 > E_0$!

4) Where does the energy comes from?
From rotation: the BH reduces its angular momentum when the particle falls

Reduction of J_H in the Penrose process

- χ^μ is future directed on the horizon.

Therefore for an observer (timelike) falling in the BH:

$$0 > p^\mu \chi_\mu = p^\mu(\xi_\mu + \Omega_H \phi_\mu) = -E + \Omega_H L$$

$L = p^\mu \chi_\mu$ so we have:

$$L < \frac{E}{\Omega_H}$$

Therefore a particle of negative energy brings negative angular momentum.

Then the (Kerr) BH experiences

$$\delta M = E \qquad \delta J = L$$

and we have:

$$\delta J < \frac{\delta M}{\Omega_H}$$

Using $M_{irr}^2 = \frac{A}{16\pi} = \frac{1}{16\pi} 8\pi\left(M^2 + \sqrt{M^2 - \frac{J^2}{M^4}}\right) = \frac{1}{2}\left(M^2 + \sqrt{M^2 - \frac{J^2}{M^4}}\right)$

one has

$$\delta J - \frac{\delta M}{\Omega_H} < 0 \iff \delta M_{irr} > 0 \quad \text{(increase of area)}$$

When we extract energy of the BH we reduce its angular momentum.

This cannot be done beyond $M = M_{irr}$, corresponding to $J = 0$

$$\boxed{M^2 = M_{irr}^2 + \frac{1}{4}\frac{J^2}{M_{irr}^2}}$$

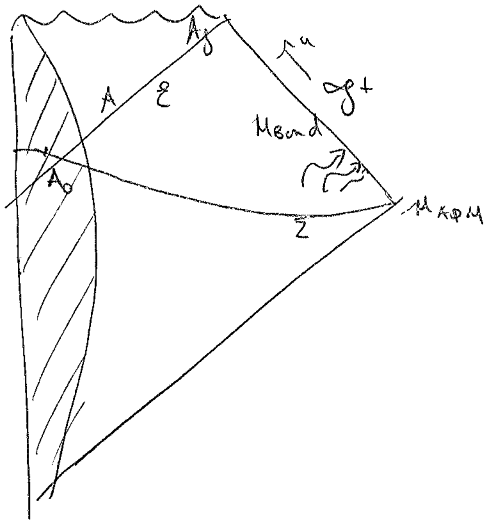
The maximum energy we can extract is $\frac{\delta M}{M} = 1 - \frac{1}{\sqrt{2}} \sim 29\%$

(the same that in the collision of two BHs using area law theorem).

Comparison to Blandford-Znajek

Field version (instead of particles): superradiant scattering: reflected waves satisfying certain criteria have larger energies than the incident ones.

Penrose conjecture $\boxed{A_0 \leq 16\pi M_{ADM}^2}$



1) Consider a 3-slice intersecting a horizon, with area A_0.
Total mass M_{ADM}

2) On the event horizon, using Hawking area $A \geq A_0$

3) Assuming final stationarity solution and using uniqueness theorems

$$A_0 \leq A \leq A_f = 8\pi\sqrt{M_f^2 - a^2} \leq 16\pi M_f^2$$

4) On $\mathcal{J}^+$: GWs takes away energy:

$$M_{Bondi} = M_{ADM} - \int du\, |N|^2 \leq M_{ADM}$$

news

$$M_f \leq M_{Bondi} \leq M_{ADM}$$

$$\boxed{A_0} \leq A \leq A_f \leq 16\pi M_f^2 \leq 16\pi M_B^2 \leq \boxed{16\pi M_{ADM}^2}$$

(If we consider AH, we must substitute A_0 by A_{min}, being A_{min} the area of the minimal surface enclosing the AH in Σ).

$$\boxed{A_0 \leq 16\pi M_{ADM}^2}$$

Problem at ID level

Leading for AHs violating this would represents a strong indication of violation of C.C.

This is a refinement of the mass positivity theorem $M_{ADM} \geq 0$ → BH spacetimes.

It includes ("test") essentially all ingredients of the standard picture of grav. collapse.