Lecture 2

April 30, 2013

Abstract

General invitation to the course.

Contents

1 Lecture 2: Gravity as spacetime curvature I: manifolds, tensors, spacetime metric

1.1 Events in *spacetime*: manifolds and coordinates

Notion of manifold to describe events in space and time. Absence of *a priori* given structures in General Relativity: all objects are fixed dynamically. Coordinates understood as *labels* without intrinsic meaning: need of coordinate independence of physical statements.

1.1.1 The manifold of physical events

Newtonian description. Let us start by considering the description of a point-like physical process happenning in space and time in the context of Newtonian physics. A basic tenet in the theory is the existece of a special class of reference frames in which Newton laws apply: these are called *inertial frames*. This provides an "a priori" structure in the theory, "rigid" in the sense that does not results from any dynamical equations. In particular, such frames provide a set of spatial coordinates (x, y, z) and a time coordinate t, permitting to asignate a coordinate time and a coordinate position to a physical "event" p, say the presence of a particle:

$$p \mapsto (t_p, x_p, y_p, z_p) \tag{1}$$ {e:ptox}

Considering a physical point-like particle to fix ideas, its evolution in time and space is described by a "trajectory" parametrized be a label λ:

$$p(\lambda) \mapsto (t_p(\lambda), x_p(\lambda), y_p(\lambda), z_p(\lambda)) \tag{2}$$

In Newtonian physics, t is a universal parameter and it is natural to use $\lambda = t$ so that $x_p(t), y_p(t), z_p(t))$ describe the trajectory. In writing down the dynamical equations, we have freedom in choosing (x, y, z) up to a Galilean tranformation: translations, rotations and boosts.

$$t' = t + t_0 \ , \ \vec{x}' = \vec{x} - \vec{a} \ , \ \vec{x}' = R(\epsilon)\cdot\vec{x} \ , \ \vec{x}' = \vec{x} - \vec{v}t \quad , \quad R(\vec{\epsilon}) \in SO(3) \tag{3}$$ {Galilean_t

Here $\vec{\epsilon}$ can be parametrised, say, by the Euler angles. As an example of a rotation, we make explicit the rotation of angle φ (Euler angle α) around the z axis, *mixing* the x and y coordinates

$$R(\varphi, 0, 0) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & & 1 \end{pmatrix} \tag{4}$$ {e:rotation

Otherwise, the coordinates (x, y, z) have a geometric content as associates to inertial frames. In particular rotations preserve the Euclidean metric in $\mathbb{R}^3$: $\mathrm{diag}(1, 1, 1)$.

Special relativity. The same reasoning essentially applies to special relativity. Although time is no longer abosolute, the notion of inertial frame exists, providing an a priori structure for the description of physical events. The freedom in the choice of $\mathbf{x} = (ct, x, y, z)$ is up to a Poincaré transformation, where time and spatial coordinates are "mixed"

$$\mathbf{x}' = \mathbf{a} + \Lambda\cdot\mathbf{x} \quad , \tag{5}$$ {e:Poincare

where matrices Λ preserve the Minkowski metric $\mathrm{diag}(-1, 1, 1, 1)$ in $\mathbb{R}^4$, spanning the Lorentz group $SO(1, 3)$. We make explicit the form of a boost along the x direction with velocity v

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\frac{v}{c}\gamma & 0 & 0 \\ -\frac{v}{c}\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \tag{6}$$

with $\gamma = \sqrt{1 - (v/c)^2}$. Noting $\gamma^2 - (v/c)^2\gamma^2 = 1$ we can write the boost matrix as

$$\begin{pmatrix} \cosh\alpha & -\sinh\alpha & 0 & 0 \\ -\sinh\alpha & \cosh\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{7}$$ {e:boost_al

where $\tanh\alpha = -v/c$. Note that, as the rotation R in (4) mixed coordinates x a y, the boost trasnformation in (7) acts as a kind of *rotation* in the (t, x) subspace. In sum, also in special relativity a geometric meaning is associated to the coordinate structure of inertial frames.

General Relativity. A basic tenet in the general relativistic description is that all structures in the theory must be determined dynamically, through the resolution of the appropriate equations. In particular, this means that the a priori notion of inertial reference frame is absent. Still, in order to have an analytical description, we need to associate to a physical event p some "labels" (t, x, y, z), as in (1). However, now the "coordinates" (t, x, y, z) are completely devoid of geometric or physical meaning. They are simply labels without intrinsic meaning and the dynamical description should be independent of them.

Physical statements must be also independent of the choice of coordinates. As an example, the coordinate description of an object trajectory has no intrinsic physical meaning. Different descriptions are possible, none of them being privileged. But the meeting of two objects along its dynamical evolution has an intrinsic physical meaning: the fact that the two trajectories cross is independent of the coordinate description. This provides an example of an spacetime "event".

[**Figure crossing of two trajectories**]

Spacetime manifold. Spacetime is the ensemble M of all physical intrinsic events. As such, M an abstract space. We require some structure on this space[1].

In particular, we require that spacetime events can be locally parametrized by formal time and space labels. That is, although the global description of M as a whole can be complicated, locally it should look like R^4: we require that M can be locally patched to open sets in R^4.

This leads to the notion of *local chart*, that is simply a way of parametrizing a open set $U \in M$ by an open set $\tilde{U}$:

$$\begin{array}{rcl} \varphi : U & \rightarrow & \tilde{U} \in \mathbb{R}^4 \\ p & \mapsto & x^\mu \equiv (x^0, x^1, x^2, x^3) \end{array} \tag{8}$$

We do not have "access" directly to p, but to its coordinate representation $x^\mu = (x^0, x^1, x^2, x^3)$. The coordinate representation has no physical/geometrical content, and different labelings are possible:

$$\begin{array}{rclcrcl} \varphi_1 : U_1 & \rightarrow & \tilde{U}_1 \in \mathbb{R}^4 & , & \varphi_2 : U_2 & \rightarrow & \tilde{U}_2 \in \mathbb{R}^4 \\ p & \mapsto & (x^0, x^1, x^2, x^3) & , & p & \mapsto & (y^0, y^1, y^2, y^3) \end{array} \tag{9}$$

so that

$$\begin{array}{rcl} \phi_2 \circ \phi_1^{-1} : \tilde{U}_2 & \rightarrow & \tilde{U}_1 \\ (x^0, x^1, x^2, x^3) & \mapsto & (y^0, y^1, y^2, y^3) \end{array} \tag{10}$$

[**Figure charts**]

In simple terms this represents a change of coordinates in the local description of M in $U_1 \cap U_2$:

$$\left\{ \begin{array}{rcl} y^0 & = & y^0(x^0, x^1, x^2, x^3) \\ y^1 & = & y^1(x^0, x^1, x^2, x^3) \\ y^2 & = & y^2(x^0, x^1, x^2, x^3) \\ y^3 & = & y^3(x^0, x^1, x^2, x^3) \end{array} \right. \tag{11}$$

{e:changeof

The spacetime st M is covered by a collection of charts (U_i, φ_i). The collection of charts is called an *atlas* of M. This confers M with the structure of a (topological) *manifold*.

[1] The first thing we should require is a topology on this set, i.e. a notion of local "neighborhoods" in M

1.2 Vectors and tensors. Lie derivative

Vectors on a manifold: tangent space. Vectors as derivations. Contravariant and covariant tensors. Tensors as local (point-like) objects on the manifold. Passive and active views of coordinates changes (diffeomorphisms): push-forward and pull-back transformations. Lie derivative as tensor variation along an infinitesimal diffeomorphism.

1.2.1 Linear approximation of the spacetime: tangent plane

We need more structure in order to manipulate efficiently the geometrical/physical objects. In particular, we want to be able to approximate the manifold M by linear structures: essentially what we do very well is linear algebra.

figure derivative

This is in the very same way that we may aproximate a non-linear function $f(x)$ at a point x_0 by its derivative

$$f(x) \sim f(x_0) + x\frac{df}{dx}(x_0) \ , \tag{12}$$

or, more generally, a non-linear application between two spaces by its differential (actually characterized as its best linear approximation).

In this sense, we want to be able to approximate M close to a given point p by a *tangent plane* T_pM. This tangent plane is provides a linear approximation to M.

1.2.2 Vectors as curve derivatives.

A first way to look at the vectors of this linear space is as the derivatives of curves passing through p. In a coordinate description[2] in coordinates $\{x^\mu\}$, the curve $\gamma : \mathbb{R} \to U$ passing through p, with $p = \gamma(0)$, is represented as

$$\begin{aligned} \gamma : \mathbb{R} \ &\rightarrow \ \varphi(U) \\ \lambda \ &\mapsto \ x^\mu(\lambda) = (x^0(\lambda), x^1(\lambda), x^2(\lambda), x^3(\lambda)) \end{aligned} \tag{13}$$

We introduce the components of a vector V at T_pM in a basis associated with coordinates $\{x^\mu\}$, as

$$V^\mu = \left.\frac{dx^\mu}{d\lambda}\right|_{\lambda=0} = \left.\begin{pmatrix} \frac{dx^0}{d\lambda} \\ \frac{dx^1}{d\lambda} \\ \frac{dx^2}{d\lambda} \\ \frac{dx^3}{d\lambda} \end{pmatrix}\right|_{\lambda=0} \tag{14}$$

In order to make this vector notion independent of the choice of coordinates, we must impose coordinate changes (11) to be differentiable. Indeed, rewriting the same curve γ in a coordinate system (y^0, y^1, y^2, y^3), and using (11) we find[3]

$$\left.\frac{dy^\mu}{d\lambda}\right|_{\lambda=0} = \left.\left(\frac{\partial y^\mu}{\partial x^\nu}\right)\right|_{\lambda=0} \left.\frac{dx^\nu}{d\lambda}\right|_{\lambda=0} \tag{15}$$

[2]An intrinsic definition of a vector not referring to particular coordinates, can be done in terms of classes of equivalence of curves

[3]This is the first encounter to the so-called index convetion of summation of repeated indices.

where the Jacobian matrix $\left(\frac{\partial y^\mu}{\partial x^\nu}\right)$

$$\left(\frac{\partial y^\mu}{\partial x^\nu}\right) = \begin{pmatrix} \frac{\partial y^0}{\partial x^0} & \frac{\partial y^0}{\partial x^1} & \frac{\partial y^0}{\partial x^2} & \frac{\partial y^0}{\partial x^3} \\ \frac{\partial y^1}{\partial x^0} & \frac{\partial y^1}{\partial x^1} & \frac{\partial y^1}{\partial x^2} & \frac{\partial y^1}{\partial x^3} \\ \frac{\partial y^2}{\partial x^0} & \frac{\partial y^2}{\partial x^1} & \frac{\partial y^2}{\partial x^2} & \frac{\partial y^2}{\partial x^3} \\ \frac{\partial y^3}{\partial x^0} & \frac{\partial y^3}{\partial x^1} & \frac{\partial y^3}{\partial x^2} & \frac{\partial y^3}{\partial x^3} \end{pmatrix} \tag{16}$$ {e:Jacobian}

provides the linear change between the basis e^x_μ and e^y_μ associated to coordinates $\{x^\mu\}$ and $\{y^\mu\}$. As we see, this requires the differentiability of the changes of charts $\phi_i \circ \phi_j^{-1}$ between any elements of the atlas U_j and U_i. This defines a *differentiable manifold*[4].

1.2.3 Vectors as derivations: directional derivatives of a function.

A useful caracterization of the vectors of the tangent space T_pM is given by a generalization of the notion of directional derivative of a function. Let us consider the coordinate representation of a function $f : M \to \mathbb{R}$, in coordinates $\{x^\mu\}$:

$$\begin{array}{rcl} f : \tilde{U}_1 & \to & \mathbb{R} \\ x^\mu & \mapsto & f(x^\mu) = f(x^0, x^1, x^2, x^3) \end{array} \tag{17}$$

Let us consider the directional derivative of f along a vector V^μ. In order to evaluate it, we consider a curve $\gamma(\lambda)$, such that $V^\mu = dx^\mu/d\lambda$ and calculate

$$\frac{df}{d\lambda} = \frac{\partial f}{\partial x^\mu}\frac{dx^\mu}{d\lambda} = \frac{\partial f}{\partial x^\mu}V^\mu = \left(V^\mu \frac{\partial}{\partial x^\mu}\right) f = V(f) \tag{18}$$

where we can denote the vector V as $V = V^\mu \partial_\mu$ can be understood as a derivation on functions.

This approach provides a natural notation for the linear basis e_μ at T_pM associated to $\{x^\mu\}$, as derivations along the coordinates x^μ

$$e^x_\mu \equiv \frac{\partial}{\partial x^\mu} \tag{19}$$

When there is no possible confusion in the coordinate basis, we will denote $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$. Therefore we can write the vector V as

$$V = V^\mu \partial_\mu \tag{20}$$

1.2.4 Tangent space T_pM and dual tangent space T^*_pM.

The dual space T^*_pM to T_pM is the set of linear applications

$$\omega : T_pM \to \mathbb{R} \tag{21}$$

Reciprocally, vectors in T_pM can be seen as linear applications

$$V : T^*_pM \to \mathbb{R} \tag{22}$$

[4] Other kind of manifolds can be considered by imposing other conditions on chart changes. For example, analytic manifolds consists in imposing analyticity on $\phi_i \circ \phi_j^{-1}$

Once the basis e^x_μ on T_pM associated with a coordinate system $\{x^\mu\}$, we can introduce the dual basis on T^*_pM, ω^μ_x, as

$$\omega^\mu_x(e^x_\nu) = e^x_\nu(\omega^\mu_x) = \delta^\mu{}_\nu \tag{23}$$

A geometric understanding of ω^μ_x comes naturally in terms of the differential of a function. Let us consider the differential df of f (as an application $f : U_1 \to \mathbb{R}$), that provides a formal infinitesimal variation df that we can denote as

$$df = \frac{\partial f}{\partial x^\mu} dx^\mu \tag{24}$$

df can be seen as an element in the dual T^*_pM by defining

$$df(V) = V(df) \equiv V(f) \tag{25}$$

Then, the differentials dx^μ can be seen to provide the basis ω^μ_x in T^*_pM, dual to e^x_μ

$$\begin{aligned}
&\omega^\mu_x = dx^\mu \\
&\omega^\mu_x(e^x_\nu) = dx^\mu \left(\frac{\partial}{\partial x^\nu}\right) = \frac{\partial}{\partial x^\nu}(dx^\mu) = \frac{\partial}{\partial x^\nu}(x^\mu) = \delta^\mu{}_\nu
\end{aligned} \tag{26}$$

Gradients, vectors, directional derivatives. A generalization of the standard gradient ∇f of a function is provided by df. Contracting the gradient with a given vector V^μ, we construct the directional derivative along V^μ. The latter is given above by $V(f) = V(df)$. It is useful to introduce a notation in terms of the "nabla" operator

$$\nabla f \;=\; df = \partial_\mu f dx^\mu = \nabla_\mu f dx^\mu \tag{27}$$

$$\nabla_V f \;=\; V(df) = V(f) = V^\mu \partial_\mu(f) = V^\mu \nabla_\mu f \tag{28}$$

where $\nabla_\mu f = \partial_\mu f$.

1.2.5 Vector and tensor fields

Up to now, all considered vectors live in a tangent space T_pM associated to a given point p in M. We can consider now the ensemble of all tangent spaces, defining the *tangent bundle* TM. Analogously one introduces the dual tangent bundle

$$TM = \bigcup_{p\in M} T_pM \quad , \quad T^*M = \bigcup_{p\in M} T^*_pM \tag{29}$$

A *vector field* V is constructed by assigning to every point $p \in M$ a vector in its tangent space

$$V : M \;\to\; TM \tag{30}$$

$$p \;\mapsto\; V_p \in T_pM \tag{31}$$

and, analogously, a *1-form* α as

$$\alpha : M \;\to\; T^*M \tag{32}$$

$$p \;\mapsto\; \alpha_p \in T_pM \tag{33}$$

Using the linear duality of TM and T^*M, vector fields and 1-forms are characterized as linear mappings

$$V : T^*M \to \mathbb{R} \quad , \quad \alpha : TM \to \mathbb{R} \tag{34}$$

This permist to introduce more general class of fields, *tensor fields* as multilinear mappings

$$T : T^*M \otimes T^*M \otimes ...^{n)} \otimes TM \otimes TM \otimes ...^{m)} TM \to \mathbb{R} \tag{35}$$

Bases for tensors can be constructed as tensor products of basis e_μ on TM and w^μ on T^*M, so that

$$\boldsymbol{T} = T^{\mu_1\mu_2...\mu_n}{}_{\nu_1\nu_2...\nu_m} \partial_{\mu_1} \otimes \partial_{\mu_2} ... \otimes \partial_{\mu_n} \otimes dx^{\nu_1} \otimes dx^{\nu_2} ... \otimes dx^{\nu_m} \tag{36}$$

The tensor $\boldsymbol{T}$, or simply $T^{\mu_1\mu_2...\mu_n}{}_{\nu_1\nu_2...\nu_m}$, is said to be n-times *contravariant* and m-times *covariant.*

Tensors as point-like and linear objects. Note that from the very construction

In addition, their linear character guarantees the following key property: If a tensor vanishes in a certain coordinates system, it vanishes in all coordinates systems

1.3 The metric tensor. Metric type of vectors.

The spacetime is more than the collection of occurring physical "events". It must be endowed with a structure capable of determining which are the spacelike directions, the timelike directions and the directions followed by light rays, as well as spatial and time *distances* between events.

Following the model of special relativity, this is accomplished by introducing an additional structure to the differentiable manifold M, namely a (non-degenerate) Lorentzian metric tensor $\boldsymbol{g}$. A spacetime is then given by the couple $(M, g_{\mu\nu})$.

1.3.1 Metric tensor

A metric tensor $\boldsymbol{g}$ is a 2-times covariant tensor

$$\boldsymbol{g} = g_{\mu\nu} dx^\mu \otimes dx^\nu \tag{37}$$

satisfying:

i) It is symmetric: $g_{\mu\nu} = g_{\nu\mu}$.

ii) It is non-degenerate: if $\boldsymbol{V}$ is such that $\boldsymbol{g}(\boldsymbol{V}, \boldsymbol{W})$, $\forall \boldsymbol{W}$, then $\boldsymbol{V} = 0$

The symmetric tensor $g_{\mu\nu}$ can be diagonalized at each point $p \in M$. If at each T_pM, a basis can be chosen such that (the non-degeneracy conditions guarantee that there are no zeros in the diagonal).

$$\boldsymbol{g}_p = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \tag{38}$$

then we say that the metric $\boldsymbol{g}$ is of Lorentzian type.

Raising and lowering of indices. The metric $g_{\mu\nu}$ provides a canonical isomorphism between TM and T^*M, not depending on coordinates. Indeed, given its non-degenerate character we consider the *metric* tensor on $T * M$ whose component expression, denoted as $g^{\mu\nu}$ is given by the inverse matrix of $g_{\mu\nu}$. That is

$$g^{\mu\rho}g_{\rho\nu} = g_{\nu\rho}g^{\rho\mu} = \delta^{\mu}{}_{\nu} \tag{39}$$

Then, given a contravariant vector V^μ and a covariant vector α_μ, we construct the associated covariant and covariant vectors, respectively, as

$$V_\mu \equiv g_{\mu\nu}V^\nu \quad , \quad \alpha^\mu \equiv g^{\mu\nu}\alpha_\nu \tag{40}$$

This operations are usually referred to as *lowering* and *raising* indices.

1.3.2 Norm of a vector, metric type and light cone.

The squared-norm of a vector is given by

$$V^2 = \boldsymbol{g}(\boldsymbol{V}, \boldsymbol{V}) = g_{\mu\nu}V^\mu V^\nu = V^\mu V_\mu \tag{41}$$

The Lorentzian nature of $g_{\mu\nu}$ permit to classify the vectors in three cathegories

i) Spacelike vectors: $g_{\mu\nu}V^\mu V^\nu > 0$.

ii) Timelike vectors: $g_{\mu\nu}V^\mu V^\nu < 0$.

iii) Lightlike or null vectors: $g_{\mu\nu}V^\mu V^\nu = 0$.

Therefore, the Lorentzian structure of the spacetime permits to introduce at each point p the notion of *light cone*, as the set of vectors in T_pM of zero norm. Light curves move along light cones in trajectories with null derivative vector. Particles moving at a speed smaller that light velocity lay inside the light cones, with timelike derivatives. Finally, particle moving faster than light, or simply curves joining points that are *simultaneous* in some coordinate system, have spacelike derivatives.

[**Figure lightcone**]

Measuring distances: element of line. The light cone structure of the spacetime allows us to structurate the spacetime in spacelike, timelike and lightlike directions. But the metric has more structure (actually very little more, just a scale), permitting us to measure distances spacelike curves and time intervals along timelike curves. This is provided by the notion of *element of line* associated to the metric in a given coordinate system, simply a quadratic form on infinitesimal displacements in spacetime:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\mu \tag{42}$$ {e:LineEleme

This can be seen as a generalization of Pythagoras theorem for infinitesinal triangles.

If we consider a spacelike curve $\gamma(\lambda)$ parametrized by λ in coordinates $\{x^\mu\}$, i.e. $(x^\mu(\lambda))$, the evaluation of (42) on $\gamma(\lambda)$ gives

$$ds^2 = g_{\mu\nu}(\gamma(\lambda))\frac{dx^\mu}{d\lambda}\frac{dx^\mu}{d\lambda}d\lambda^2 \quad . \tag{43}$$

For a spacelike curves the *arc length* can be simply written as

$$ds = \sqrt{g_{\mu\nu}(\gamma(\lambda))\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}d\lambda \tag{44}$$

With our convention for the spacetime signature $(-1, 1, 1, 1)$, the element of proper time along timelike curves is given by $-c^2 d\tau = ds^2$, that is

$$d\tau = \frac{1}{c}\sqrt{\left|g_{\mu\nu}(\gamma(\lambda))\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\right|}d\lambda \tag{45}$$

{e:dtau}

1.3.3 Observers.

An observer in General Relativity is provided by a timelike curve γ whose 4-velocity u^μ is normalized to -1, that is

$$u^\mu = \frac{d^\mu}{d\lambda} \quad , \quad u^\mu u_\mu = g_{\mu\nu}(\gamma(\lambda))\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} \quad = -1 \tag{46}$$

Using (45) we can write $u^\mu = \frac{dx^\mu}{d\tau}$.

1.4 Minkowski spacetime. Rindler coordinates

The first spacetime we have encountered corresponds to the one in special relativity, corresponding to the absence of gravity. Its line element in coordinates corresponding to an inertial frame

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2 \tag{47}$$

{e:Minkowsk

Note that Poincaré trasnformations (5) preserve the form of this line element. They are the first example of isometries. The Minkowski geometry illustrates some of the points in this lecture. First, note that parametrizing a timelike curve by $\lambda = ct$, proper time writes

$$d\tau = \sqrt{1 - \frac{1}{c^2}\frac{d\vec{x}}{dt}\cdot\frac{d\vec{x}}{dt}}dt \tag{48}$$

and an observer

$$u^\mu = \left(\gamma, \gamma\frac{d\vec{x}}{dt}\right) \tag{49}$$

with $\gamma = dt/d\tau = 1/\sqrt{1 - \frac{1}{c^2}\frac{d\vec{x}}{dt}\cdot\frac{d\vec{x}}{dt}}$.

1.4.1 Absence of geometric meaning in the coordinates

We have insisted in the absence of a priori geometric meaning in the coordinates in General Relativity. This is illustrated byy the following example. Consider the metric with line element

$$ds^2 = -\frac{1}{t^2}dt^2 + dx^2 + dy^2 + dz^2 \tag{50}$$

for $0 < t\infty$, $-\infty < x < \infty$, $-\infty < y < \infty$, $-\infty < z < \infty$. This suggests a metric with a bad behaviour as one approaches $t = 0$, possibly indicating some geometric non-trivial behaviour in its vicinity. However, if we make the transformation of variables

$$t' = \ln t \ , \ x' = x \ , \ y' = y \ , \ z' = z \tag{51}$$

with $0 < t'\infty$, $-\infty < x' < \infty$, $-\infty < y' < \infty$, $-\infty < z' < \infty$. we realize that the metric can be written as

$$ds^2 = -c^2 dt'^2 + dx'^2 + dy'^2 + dz'^2 \tag{52}$$

where we recognize the familiar Minkowski spactime. We conclude that coordinates (t, x, y, z) are just *labels* without any intrinsic meaning.

Remarks.

- i) The expression *choice of coordinates* actually refers to the freedom in choosing the functional form of four of the functions in the set of ten functions $g_{\mu\nu}(x)$, where $\{x^\mu\}$ are just formal labels without meanings.

  In general, for a given metric the remaining six functions cannot be freely chosen (the freedom in coordinate choice is exhausted). In particular the question of under which conditions a line element can be trasnformed to the form (47), see **[Exercise ...]** implies the resolution of an overdetermined system of partial differential equations, so that in general has no solution. Only in special cases satisfying certain *integrability conditions* the system can be solved. As we will comment later, this integrability conditions are given in terms of a tensorial quantity, precisely the *curvature tensor*. In other words, the gravitaional field.

- ii) The singular behaviour in the metric functions can be due to two reasons: a) an actual singularity in the metric, b) a pathologic behaviour of the coordinates. Deciding with is the case is not always easy. The Rindler metric in **[Exercise ...]** provides a paradigmtic example of this that illustrates the behaviour that we will find in black holes.

1.5 Exercises: tensor manipulation (indices gymnastics).

- Transformation rules of contravariant and covariant vectors under a coordinate transformation.
- Transformation of the metric tensor.
- Transdormation of the volume element.
- Coordinate velocity: Is the coordinate velocity of light constant?
- Conformal structure and light cone structure: conformal transformations of the spacetime metric.