2019/20

Master Mathematical Physics (Dijon) - General Relativity
Exam, 2nd Session
3 July (3 hours)

Every student commits, by participating in this exam, to respect the rules at the Université de Bourgogne, namely to answer without any external help to the posed questions.

COMPULSARY SIGNATURE:

1. Given the metric on the sphere $\mathbb{S}^2$, defined by the line element (R is a constant):

$$ds^2 = R^2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2\right) \ ,$$

calculate the Ricci scalar R by using the Cartan's structure equations.

2. i) Consider (2-dim) Minkowski spacetime and timelike connected points p and q. Justify that a non-accelerated observer maximizes the proper time between p and q.

   ii) Justify that there exist observers travelling from p to q that make the ellapsed proper time as small as desired (that is, there is no time-like curve that minimizes the proper time between p and q).

   iii) Consider now Schwarzschild spacetime with mass M and two observers at fixed (θ, φ), initially at $r_1 > 2M$. At some given time, one of the observers descends to r_2, with $2M < r_2 < r_1$. After a very long time, this observer comes back to r_1. Who is older when the two observers meet? Compare with the situation in i): which physical principle can be invoked to establish an analogy between the situations i) and iii)?

   iv) We consider again these two observers in Schwarzschild. As before, one of them stays at a fixed (θ, φ) at a given r, but now the second one follows a circular orbit without using the engines. Without making any calculation, just with a reasoning based on the understanding of i) and ii): which one will be older when they meet after one complete orbit?

3. Given a curve timelike (or spacelike) γ joining two points p and q, their distance ℓ along γ is given by

$$\ell = \int_p^q \sqrt{\mp g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} d\lambda \ , \qquad (1)$$

where $x^\mu = x^\mu(\lambda)$ is the coordinate expression of γ in a local chart and $\dot{x}^\mu = \dfrac{dx^\mu}{d\lambda}$.

Applying a variational principle (with Lagrangian $L(x, \dot{x}) = (\mp g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu)^{\frac{1}{2}}$), show that curves for which ℓ is an extremum satisfy

$$\ddot{x}^\mu + \Gamma^\mu_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta = \kappa(\lambda)\dot{x}^\mu \ , \qquad (2)$$

with $\kappa = \dfrac{d \ln L}{d\lambda}$.

4. Derive the expression of the gravitational redshift for two observers situated at different heights in the (vacuum) exterior of a spherically symmetric distribution of matter of total mass M.