2018/19

Master Mathematical Physics (Dijon) - General Relativity

Exam

17 May (3 hours)

1. i) The following expressions hold for a metric $\boldsymbol{g} = g_{\mu\nu} dx^\mu \otimes dx^\nu$ in a 2-dim manifold:

$$R^\mu{}_{\nu\rho\sigma} \;=\; \frac{R}{2}\left(\delta^\mu{}_\rho g_{\nu\sigma} - \delta^\mu{}_\sigma g_{\nu\rho}\right) \ ,$$

where $R^\mu{}_{\nu\rho\sigma}$ is the Riemann tensor and R is the Ricci scalar. Determine the value of α in the expression:

$$R_{\mu\nu} = \alpha R g_{\mu\nu} \ .$$

ii) Given now the metric on $\mathbb{R}^2_+$, defined by (H is a constant):

$$\boldsymbol{g} = \frac{H}{y^2}\left(-dt \otimes dt + dy \otimes dy\right) \ ,$$

calculate the Riemann tensor $R^\mu{}_{\nu\rho\sigma}$, the Ricci tensor $R_{\mu\nu}$, the Ricci scalar R and verify the result in i).

2. Given the Schwarzschild metric in standard coordinates (t, r, θ, φ), consider the change of variables:

$$\begin{cases} t' = t + 2M \ln\left(\dfrac{r}{2M} - 1\right) \\ r' = r \\ \theta' = \theta \\ \varphi' = \varphi \end{cases}$$

i) Write the line element in the coordinates $(t', r', \theta', \varphi')$.

ii) Consider the radial outgoing and ingoing null trajectories (i.e. θ' and φ' constant). Determine the ODEs satisfied by these trajectories and sketch the corresponding outgoing and ingoing curves in a (t', r') diagram, in particular showing the null cones.

iii) Determine the proper time $\Delta\tau$ and coordinate time $\Delta t'$ between the emission of an ingoing radial light ray from an observer at position r_+ and its reception at r_- (with $r_+ > r_-$). If $r_- = 2M$, what can be concluded about the new coordinate system as compared with the original one?

3. i) The spacetime metric around a spherically symmetric planet is given by the Schwarzschild metric. We consider two astronauts in a spaceship orbiting the planet at a certain height. At a given time, one of them goes down (fastly, in a time scale δT_1) to the planet and stays there for a long time T (namely $T \gg \delta T_1$). After this time, he comes back (again fastly, δT_2 with $T \gg \delta T_2$) to the spaceship and meets the astronaut who stayed there. Justify which astronaut is younger at the moment of meeting again.

ii) Given the metric on $\mathbb{R}^3$:

$$\boldsymbol{g} = f^2(z)\left(dx \otimes dx + dy \otimes dy\right) + dz \otimes dz \ ,$$

with $f(z) > 0$, justify that $\boldsymbol{g}$ is conformally flat.