2019/20

Master Mathematical Physics (Dijon) - General Relativity

Exam

8 juin (3 hours)

Every student commits, by participating in this exam, to respect the rules at the Université de Bourgogne, namely to answer without any external help to the posed questions.

COMPULSARY SIGNATURE :

1. Given the metric on $\mathbb{R}^2_+$, defined by the line element (H is a constant):

$$ds^2 = \frac{H}{y^2}\left(-dt^2 + dy^2\right) \ ,$$

calculate the Ricci scalar R.

(Privilege the use of Cartan's structure equations; if not, use any method you know).

2. i) Consider (2-dim) Minkowski spacetime and timelike connected points p and q. Justify that a non-accelerated observer maximizes the proper time between p and q.

ii) Justify that there exist observers travelling from p to q that make the ellapsed proper time as small as desired (that is, there is no time-like curve that minimizes the proper time between p and q).

iii) Consider now Schwarzschild spacetime with mass M and two observers at fixed (θ, φ), initially at $r_1 > 2M$. At some given time, one of the observers descends to r_2, with $2M < r_2 < r_1$. After a very long time, this observer comes back to r_1. Who is older when the two observers meet? Compare with the situation in i): which physical principle can be invoked to establish an analogy between the situations i) and iii)?

iv) We consider again these two observers in Schwarzschild. As before, one of them stays at a fixed (θ, φ) at a given r, but now the second one follows a circular orbit without using the engines. Without making any calculation, just with a reasoning based on the understanding of i) and ii): which one will be older when they meet after one complete orbit?

3. Consider the spacetime $\mathbb{R}\times]-1,1[$ with line element given by

$$ds^2 = -(1-y^2)dt^2 + (1-y^2)^{-1}dy^2 \ .$$

Change the line element from (t, y) to (τ, y) coordinates using

$$\tau = t + \frac{1}{2}\ln(1-y^2) \ .$$

In the (τ, y) coordinates write explicitly the Klein-Gordon equation (m constant)

$$\left(\Box - m^2\right)\phi = 0.$$

4. Given a time-orientable Lorentzian manifold, let us consider (at a given point) two vectors u^a and v^a which are timelike or null. If u^a is future directed show:

   i) $g_{ab}u^a v^b < 0$ if and only if v^a is future-oriented.

   ii) $g_{ab}u^a v^b = 0$ if and only if u^a and v^a are null and collinear.

   iii) $g_{ab}u^a v^b > 0$ if and only if v^a is past-oriented.