Geometry and physics of black holes

(Lecture notes, under work)

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An invitation to General Relativity and gravitational collapse

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1.1 A first glimpse into to Gravity as spacetime curvature

1.1.1 Robust elements in General Relativity

General Relativity explains gravity in terms of curvature of the spacetime. In particular, it provides a prescription for the determination of such curvature in terms of the presence of mass and energy in the spacetime, through the so-called Einstein equations.

Beyond the specific details of the theory, we can point out several conceptual elements that the theory teaches us and that should survive in any theory extending or substituting General Relativity. Among these elements we make explicit the following ones:

i) *Gravitational redshift*: light propagating through a gravitational field experiences a displacement in its frequency, in particular shifting to longer wavelengths when passing from stronger to weaker gravitational fields (an inverse "blue-shift" effect happens when propagating towards strong gravitational fields). Such an effect should survive General Relativity since it does not depend on the detailed form of the field equations.

ii) *Gravitational waves*: the theory presents local dynamical degrees of freedom (for spacetime dimensions ≥ 4) associated with the gravitational field, which are absent in the previous non-relativistic (Newtonian) theory. Such degrees degrees of freedom can be interpreted in terms of *dynamical tidal fields*. In this sense, the understanding of gravity in terms of the tidal deformation of objects acquires a key role in the very development of the theory.

iii) *Frame dragging*: a rotating objects "pulls" the spacetime with it, making all nearby objects experience a "force" entailing them to rotate. Again, this is independent of the detailed form of the field equations, and refers essentially to some kind of behaviour of spacetime as an *elastic medium* that cannot be detached from the material sources creating it.

All these three effects can be referred as *kinematical* in the sense that qualitatively they are independent of the particular form of the field equations for the gravitational field and mainly rely on the structural fact of constructing the theory on a curved spacetime Lorentzian manifold[1].

Special relativity offers a description of relativistic motion in the case that gravity can be neglected. In this section we describe the tension existing between special relativity and the incorporation of gravity in the picture, ultimately leading to the notion of a curved spacetime:

$$\left.\begin{array}{c}\text{Special Relativity}\\ \text{Gravity}\end{array}\right\}\text{Tension}\longrightarrow\text{Spacetime curvature}$$

The main line of reasoning is that the marriage between light propagation and gravity implies the existence of a gravitational redshift effect, and that the latter is incompatible with special relativity, leading to the notion of an intrinsically curved spacetime:

$$\left.\begin{array}{c}\text{Gravitational Redshift}\\ \text{Flat spacetime}\end{array}\right\}\text{Tension}\longrightarrow\text{Spacetime curvature}$$

We follow essentially the discussion in [5].

1.1.2 Gravitational redshift from energy conservation

Let us start by reviewing the original Einstein argument, based on a physical reasoning (namely energy conservation), leading to the existence of a gravitational redshift.

We dwell here in a Newtonian description of gravity. Let us consider a particle of mass m at a height L in a constant gravitational field (with g the module of the gravitational acceleration, so $g > 0$).

i) Initially the particle is at A and its "rest energy" (special relativity) is:

$$E^A = mc^2 \quad . \tag{1.1}$$

ii) It falls to B, having a "rest" plus "kinetic energy":

$$E^B = mc^2 + mgL \quad . \tag{1.2}$$

iii) At B, the particle is annihilated producing a photon with (the same) energy:

$$E^B_{ph} = mc^2 + mgL \quad . \tag{1.3}$$

Then the photon goes back upwards to A. If the energy of the photon at A were $E^A_{ph} = E^B_{ph} = mc^2 + mgL$, then we are able to *create* energy that we can (*immediately*) use. Indeed,

[1] These three elements are captured, in a linearized version, in the so-called *"optical" scalars* that encode the "expansion", "shear" and "twist" response of an extended body to the presence of gravitational field, see e.g. [8].

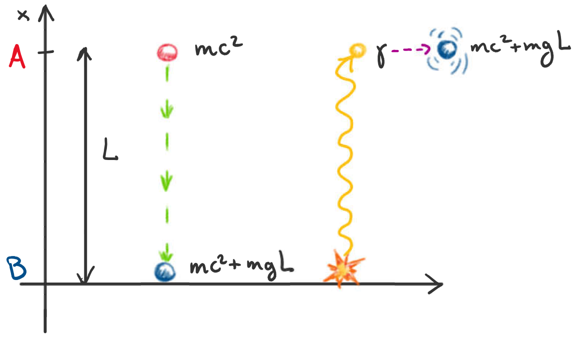

Figure 1.1: Violation of energy conservation if photons are not gravitationally redshifted.

the photon at A can be transformed into a particle of mass m and some additional energy (thermal, kinetic...) (see Fig. 1.5.1):

$$E^A = mc^2 + mgL \quad . \tag{1.4}$$

We can repeat the process n times after which we have at A a particle of mass m and a production of extra energy

$$E^A = mc^2 + nmgL \quad , \tag{1.5}$$

producing an arbitrarily large violation of the energy conservation.

The way out is to accept that the photon loses energy when going from B to A: the photon has to climb the gravity potential as a massive particle would have to. Therefore starting from B with an energy E^B_{ph} it arrives at A with an energy E^A_{ph}:

$$E^B_{ph} = mc^2 + mgL = mc^2(1 + \frac{gL}{c^2}) \quad \to \quad E^A_{ph} = mc^2 \tag{1.6}$$

Now Einstein's argument incorporates another piece of physical reasoning. In particular, at this point one uses the relation between energy of a photon and its wavelength given by quantum theory, namely

$$E_{ph} = h\nu = \hbar\omega \quad . \tag{1.7}$$

Then, using $\lambda = c/\nu$ and the redshift factor z introduced as

$$z = \frac{\lambda_A - \lambda_B}{\lambda_B} \quad , \quad 1 + z = \frac{\lambda_A}{\lambda_B} \quad , \tag{1.8}$$

one gets

$$1 + z = \frac{\lambda_A}{\lambda_B} = \frac{\nu_B}{\nu_A} = \frac{h\nu_B}{h\nu_A} = \frac{E_B}{E_A} = (1 + \frac{gL}{c^2}) \quad , \tag{1.9}$$

and therefore

$$z = \frac{gL}{c^2} \quad . \tag{1.10}$$

This expression for the redshift of a photon "going up" a gravitational field, *deduced* by Einstein in 1911 using this chain of heuristic physical arguments, would be indeed experimentally confirmed only in 1959 by Pound & Rebka [9].

1.1.3 Gravitational redshift and the principle of equivalence

The previous discussion of the gravitational redshift is physically inspiring, but can be criticized on consistency grounds. The discussion can be recast in a more systematic ("first-principles") form in terms of the key ingredient in the process of the *geometrization* of the gravitational field: the *equivalence principle*. In its more basic form it states:

"All effects of a uniform gravitational field are identical to the effects of a uniform acceleration of the coordinate system."

This is a generalization of the simple remark in the context of Newtonian particle dynamics, where we can write (we assume here equality between the inertial and gravitational mass, in order to simplify the argument)

$$m\frac{d^2x}{dt^2} = F = -mg \quad \Longleftrightarrow \frac{d^2x}{dt^2} + g = 0\;; \frac{d^2x'}{dt^2} = 0 \quad , \tag{1.11}$$

with

$$x' = x + \frac{1}{2}gt^2 \ . \tag{1.12}$$

We note that the coordinate system associated to x' moves with a uniform acceleration $a = -g$ as described by the coordinate system x. In particular, the coordinate x (in the non-accelerated system) corresponding to the center of coordinates of the accelerated system, i.e. $x' = 0$, is

$$x = x' - \frac{1}{2}gt^2 = \frac{1}{2}at^2 \quad ; \quad \text{with } a = -g \quad . \tag{1.13}$$

We say that the reference system associated with x' is in *free fall* and we see how a local gravitational force disappears for a free-falling observer. On the other hand, if we take the perspective of x' as the fundamental one, then there is no gravitational force and the force that x experiences is of *inertial* nature, as a consequence of his ("upwards") acceleration with $a = g$ from Eq. (1.12), with respect to x'. This dual vision between *inertial* and *gravitational* forces will be the key to explain test-particle motion in general relativity in terms of geodesics, that correspond to *free falling* observers.

Coming back to our discussion on the red-shift, the key element here[2] is the extension of the validity of the *equivalence principle* statement to ALL possible effects, this including electromagnetic ones, in particular light propagation.

Let us consider again the points A and B above, standing in a constant gravitational field. At a given moment, a photon γ is emitted from A to B. According to the equivalence principle we can consider an equivalent description from the perspective of a free falling observer (the system x' above), from whose perspective there is no gravitational field but instead the emitter A and receiver B suffer an upwards acceleration $a = g$, as expressed in (1.12). We can think of the "apparent" gravitational field experienced in an accelerated rocket or elevator, in absence of a gravitational source, cf. Fig.1.1.3

That is, described in the free-falling reference system[3] points A and B corresponding to

[2] In a manner analogous to the role of the *relativity principle* that leads to special relativity, where ALL physical experiments must render the same results for inertial observers.

[3] Note that, according to the equivalence principle, x' stands as a perfectly valid *inertial observer* (she/he is *not* accelerated! So, along its *free fall*, she/he follows a *straight line*) and simply perceives the emitter A and the receiver B as accelerating upwards. This perspective is the one that will be taken in general relativity, where the notion of *free fall* can be given a primitive meaning. Accordingly, in this (local) inertial system the speed of light is c.

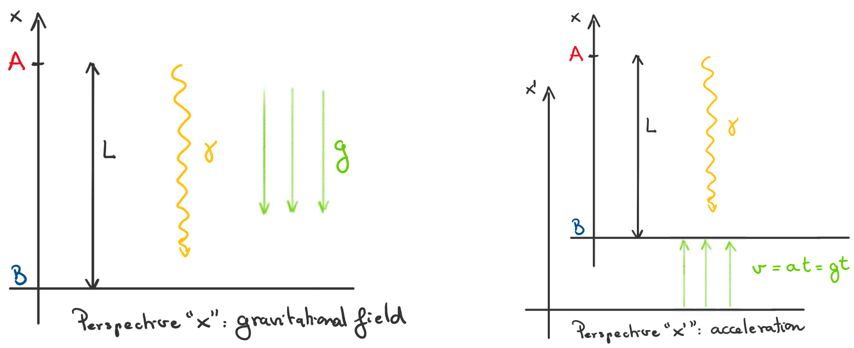

Figure 1.2: Photon moving in a constant gravitational field or, equivalently, in an accelerated frame.

emitter and observer move in a uniformly accelerated motion with $a = g$ as (note Eq. (1.12))

$$x'_A = L + \frac{1}{2}gt^2 \quad ; \quad x'_B = \frac{1}{2}gt^2 \tag{1.14}$$

i) The photon is sent from A at $t = 0$, so that B receives it at $t = t_1$. The travelled distance is[4] (light propagates in this "inertial" coordinate system at speed c and all calculations in such inertial frame are standard, only "changes" to other frames need to be adapted to special relativistic rules)

$$x'_A(0) - x'_B(t_1) = ct_1 \quad , \quad L - \frac{1}{2}gt_1^2 = ct_1 \ . \tag{1.15}$$

ii) A second photon (or the next crest in a trainwave) of is sent from A at $t = \Delta\tau_A$ and B receives it a time $\Delta\tau_B$ after receiving the first photon, that is at $t_2 = t_1 + \Delta\tau_B$. The distance traveled by the second photon is

$$x'_A(\Delta\tau_A) - x'_B(t_2) = c(t_2 - \Delta\tau_A) = c(t_1 + \Delta\tau_B - \Delta\tau_A) \ . \tag{1.16}$$

The left hand side can be re-expressed as

$$\begin{aligned} x'_A(\Delta\tau_A) - x'_B(t_2) &= x'_A(\Delta\tau_A) - x'_B(t_1 + \Delta\tau_B) = L + \frac{1}{2}g(\Delta\tau_A)^2 - \frac{1}{2}g(t_1 + \Delta\tau_B)^2 \\ &= L + \frac{1}{2}g(\Delta\tau_A)^2 - \frac{1}{2}gt_1^2 - gt_1\Delta\tau_B - \frac{1}{2}g\Delta\tau_B^2 = L - \frac{1}{2}gt_1^2 - gt_1\Delta\tau_B + O(\Delta\tau^2) \end{aligned} \tag{1.17}$$

Neglecting second-order terms in $\Delta\tau$'s, we can write

$$L - \frac{1}{2}gt_1^2 - gt_1\Delta\tau_B \approx c(t_1 + \Delta\tau_B - \Delta\tau_A) \ . \tag{1.18}$$

Subtracting (1.15) from (1.18) we get

$$-gt_1\Delta\tau_B = c(\Delta\tau_B - \Delta\tau_A) \Leftrightarrow \Delta\tau_A = \Delta\tau_B(1 + \frac{gt_1}{c}) \ . \tag{1.19}$$

Finally, approximating at first order from (1.15), $t_1 \approx \frac{L}{c}$ we get

$$\Delta\tau_A = \Delta\tau_B(1 + \frac{gL}{c^2}) \ . \tag{1.20}$$

[4] We will neglect second order terms (such as $\left(\frac{v}{c}\right)^2$ or $\left(\frac{gL}{c^2}\right)^2$) in the following discussion.

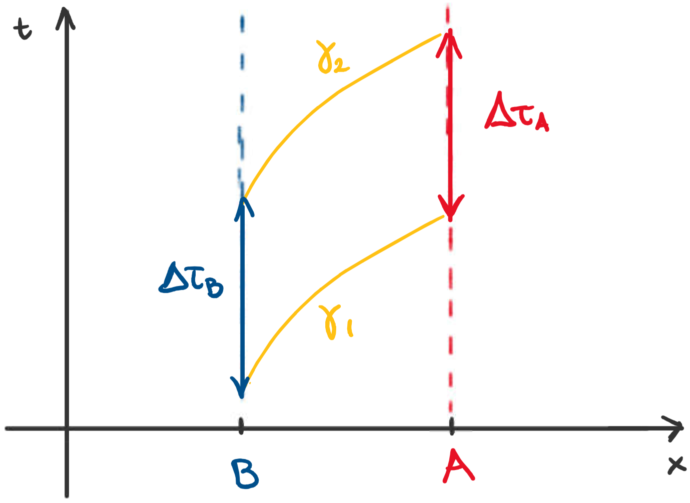

Figure 1.3: Diagram for Schild's argument on the incompatibility between gravitational redshift and flat spacetime.

iii) Now, expressing the time intervals $\Delta\tau$'s in terms of frequencies, $\Delta\tau = 1/\nu$ we write

$$\nu_B = \nu_A(1 + \frac{gL}{c^2}) \quad , \tag{1.21}$$

from where, again

$$1 + z = \frac{\lambda_A}{\lambda_B} = \frac{\nu_B}{\nu_A} = (1 + \frac{gL}{c^2}) \quad , \tag{1.22}$$

and

$$z = \frac{gL}{c^2} \quad . \tag{1.23}$$

as in Eq. (5.25).

1.1.4 Gravitational redshift implies curvature of spacetime

The previous discussions have led us to the notion that light propagating in a gravitational field gets redshifted. We can accept this either from Einstein's physical argument, or as a consequence of the equivalence principle, or simply as an experimental fact from Pound & Rebka experiment.

On the other hand, special relativity has already shown that a consistent description of particle kinematics and electrodynamics involves a spacetime perspective on space and time. Space and time are recast in a single geometric structure modeled as a linear space endowed with a flat metric of Lorentzian type: the Minkowski spacetime. At this point we show, following an argument of Schild (see Fig. 1.1.4), that the presence of a gravitational redshift is incompatible with the existence of a *flat* spacetime like in special relativity. Schild's argument is independent of the detailed mathematical description of the gravitational field. Only stationarity plays a key role in the argument. Let us consider two observers A and B at rest one with respect to the

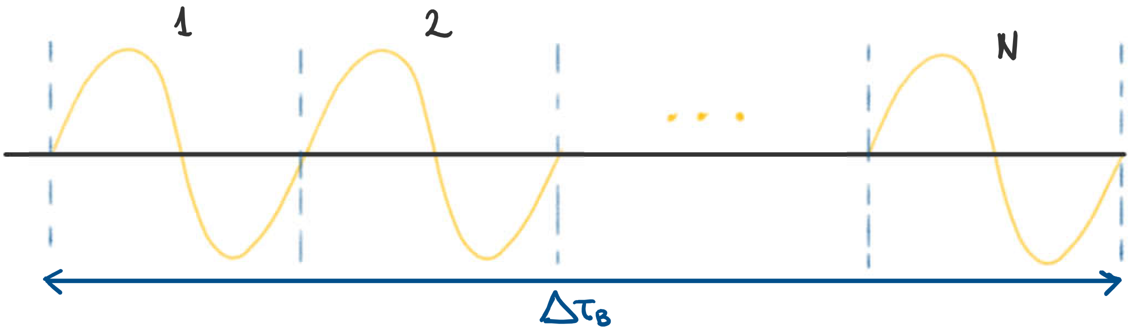

Figure 1.4: Wave train of signals emitted from B towards A.

other and with respect to the Earth (namely, the source of the gravitational field). Whatever the nature of the gravitational field is, it will present a stationary configuration.

At some given time, a signal is emitted from B towards A. Let us assume that it is a periodic signal with N cycles. Then (see Fig. 1.1.4)

$$N = \nu_B \Delta\tau_B \quad , \tag{1.24}$$

with ν_B the frequency and $\Delta\tau_B$ the elapsed time of the signal.

The receiver at A receives the N cycles in a time $\Delta\tau_A$, so that

$$N = \nu_A \Delta\tau_A \quad , \tag{1.25}$$

and

$$\nu_A \Delta\tau_A = \nu_B \Delta\tau_B \quad . \tag{1.26}$$

According to previous discussions, if a redshift is present we have $\nu_B > \nu_A$ and, as a consequence

$$\Delta\tau_A > \Delta\tau_B \quad . \tag{1.27}$$

However, since the gravitational field is static and the observers do not move, trajectories γ_1 and γ_2 of the respective photons must be *congruent* curves, i.e. γ_1 and γ_2 are the sames curves except from their *positions* in the space-time picture. If such curves are placed in a *flat* space and time diagram (namely, the spacetime), they must form a parallelogram, so that

$$\Delta\tau_A = \Delta\tau_B \quad , \tag{1.28}$$

in contradiction with (1.27). This contradiction indicates that the flat spacetime of special relativity, namely Minkowski spacetime, is not adequate for the description of gravity (if we want to make it compatible with the existence of gravitational redshift). If we want to stick to the spacetime vision of space and time provided by special relativity, then we must renounce to spacetime flatness. In particular, initially *parallel* light trajectories can start converging and diverging, in general *bending* in a curved spacetime. More generally, in this geometric spacetime perspective the presence of a gravitational field is realised through the curvature of spacetime. General Relativity provides a definite self-consistent manner of introducing physical sources to this spacetime curvature, namely through energy and stress of matter. At the same time, it endows this spacetime curvature, namely the gravitational field, with specific dynamics.

1.1.5 Towards the fixing of curvature: gravitational field and tides

As we have just seen, the existence of a gravitational redshift together with the notion of spacetime structure inherent to special relativity, leads to the need of *curvature* of such spacetime geometry. But the reasoning gives no clues about the manner of fixing such a curvature. On the other hand, the equivalence principle amounts to the introduction of infinitesimal (point-like) observers that do not experience gravitational forces. Two questions emerge therefore for the previous analysis:

i) How can be determined the curvature of spacetime?

ii) Is there an alternative way in which a *free falling* observer can detect the presence of a gravitational field?

Remarkably the notion of *tide* provides an approach towards both these questions. This leads us to revisit some features of Newtonian gravity.

Newtonian gravity: particle and field equation

Particle dynamical equation. In Newtonian physics, given an inertial reference system, equations of motion for particles with respect to a universal time t are given by Newton's second law

$$m_i \frac{d^2\vec{x}}{dt^2} = \vec{F} \ , \tag{1.29}$$

where m_i is the so-called inertial mass and $\vec{F}$ is a given force. Dynamics are complemented by a specific prescription of the force $\vec{F}$. In the case of the gravitational force $\vec{F_G}$ exerted by a point-like object of (active gravitational) mass M on an object of (passive gravitational) mass m_g, the force $\vec{F_G}$ is given by Newton's universal gravitational force

$$\vec{F_G} = -G\frac{Mm_g}{r^2}\hat{e}_r \ , \tag{1.30}$$

where $r = |\vec{x} - \vec{x'}|$ and $\hat{e}_r = \frac{\vec{x}-\vec{x'}}{|\vec{x}-\vec{x'}|}$, where $\vec{x}$ is the position of the mass m_g and $\vec{x}'$ is the position of mass M. Combining (1.29) and (1.30) and using (another version of) the *weak equivalence principle*, namely $m_i = m_g$, we find

$$\frac{d^2\vec{x}}{dt^2} = -G\frac{M}{r^2}\hat{e}_r \ . \tag{1.31}$$

The key point to underline here is that the acceleration of a particle in a gravitational field is completely independent of the nature of that particle, only depending on the mass of the particle creating the field $\vec{E_G}$. It is a *geometric* feature in the sense that all particles follow the same trajectories: it is therefore a property of the *background geometry*. General relativity will take this to a foundational level: it is in this sense that the theory is fundamentally *geometric*, and not in the sense of its capability to be formulated in a *covariant way* (coordinate independent).

Field equation. A important conceptual step towards general relativity is the introduction of the notion of field. In particular, the gravitational field $\vec{E}_G$ created by a particle of mass M placed at $\vec{x}'$ on a arbitrary point $\vec{x}$ is given by

$$\vec{E}_G = -G\frac{M}{r^2}\hat{e}_r \ , \tag{1.32}$$

so that when placing a particle of mass m in this field $\vec{E}_G$ it experiences a force

$$\vec{F}_G = m\vec{E}_G \ . \tag{1.33}$$

The field $\vec{E}_G$ can be written in terms of a gravitational potential ϕ as

$$\vec{E}_G = -\vec{\nabla}\phi \ , \tag{1.34}$$

where

$$\phi = -G\frac{M}{r} \ . \tag{1.35}$$

Newton's prescription (1.30) for the gravitational force can be recast in terms of an equation for ϕ. For this, consider a continuous distribution of mass with mass density $\rho(\vec{x})$ in a region D. From (1.32) we can write

$$\vec{E}_G(\vec{x}) = -G\int_D \rho(\vec{x}')\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}d^3x' \ . \tag{1.36}$$

If we now calculate the divergence $\vec{\nabla}\cdot\vec{E}$, and use

$$\vec{\nabla}\cdot\left(\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}\right) = 4\pi\delta(\vec{x}-\vec{x}') \tag{1.37}$$

we obtain

$$\vec{\nabla}\cdot\vec{E} = -4\pi G\rho(\vec{x}) \ . \tag{1.38}$$

On the other hand, taking the divergence in (1.34) we obtain $\vec{\nabla}\cdot\vec{E} = -\Delta\phi$. Finally, we can write

$$\Delta\phi = 4\pi G\,\rho(\vec{x}) \tag{1.39}$$

This is Poisson's equation, that we have obtained from Newton's expression for the gravitational force. In other direction, if we consider a pointlike source with density $\rho(\vec{x}) = M\,\delta(\vec{x}-\vec{x}')$, we can solve[5] Poisson's equation (1.39) to obtain (1.35) and therefore Newton's law (1.30) through (1.34) and (1.33). In this sense, Newton's force and Poisson's equation are equivalent. At the Newtonian level we can take the perspective we prefer, at the relativistic level Poisson's expression will be the natural starting point.

[5]Use

$$\Delta\frac{1}{|\vec{x}-\vec{x}'|} = -4\pi\delta(\vec{x}-\vec{x}') \ . \tag{1.40}$$

Note that, consistently with (1.34) and (1.36) we can write

$$\phi(\vec{x}) = -G\int_D \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}d^3x' \ , \tag{1.41}$$

so that Poisson's equation follows directly from the application of (1.40).

Tides and free fall. As we have discussed in section 1.1.3, a pointlike observer in free-fall is not able to tell about the presence of a gravitational field: its acceleration vanishes. Generalizing slightly the discussion above around Eq. (1.12), let us consider a gravitational potential $\phi = \phi(\vec{x})$ and a particle whose instantaneous position in an *inertial* reference system is $\vec{x} = \vec{x}_o$. Noting from the previous discussion $\vec{a} = \frac{d^2\vec{x}}{dt^2} = -\nabla\phi(\vec{x})$, we can make the change of coordinates

$$\vec{x}' = \vec{x} - \vec{x}_o - \frac{1}{2}\vec{g}t^2 \ , \tag{1.42}$$

with $\vec{g} = -\nabla\phi(\vec{x}_o)$ (note $\vec{x}$ is evaluated at $\vec{x}_o$ at this gradient). Taking second time derivatives we find

$$\vec{a}' = \vec{a} - \vec{g} = -\nabla\phi(\vec{x}) + \nabla\phi(\vec{x}_o) \ , \tag{1.43}$$

which vanishes at x_o. So a *pointlike* particle cannot tell if it is falling, by what she experiences at that point.

But there is a manner of telling, if one looks to another closely falling observer, separated at a distance $\vec{\ell}$. Indeed, we can evaluate

$$\vec{a}(\vec{x} + \vec{\ell}) = \vec{a}(\vec{x}) + \vec{\ell} \cdot \nabla\vec{a}(\vec{x}) + o(\vec{\ell}) \ . \tag{1.44}$$

Neglecting orders higher than the linear one, we find that the difference $\delta\vec{a} = \vec{a}(\vec{x} + \vec{\ell}) - \vec{a}(\vec{x})$ of accelerations satisfy (using $\vec{a} = \vec{E}_G = -\nabla\phi$)

$$\delta a_i = \ell^j \nabla_j (E_G)_i = -\ell^j \frac{\partial^2\phi}{\partial x^j \partial x^i} = -\ell^i \mathcal{E}_{ji} \ , \tag{1.45}$$

where $\mathcal{E}_{ij}$

$$\mathcal{E}_{ij} = \frac{\partial^2\phi}{\partial x^i \partial x^j} \tag{1.46}$$

is the so-called *tidal* tensor, corresponding indeed to the gradient of the gravitational field (tides). The tidal acceleration δa_i is non-local, it depends linearly on the separation $\vec{\ell}$ between free-falling observers, but the tidal tensor field is indeed local. We make two remarks:

i) In contrast with the field $\vec{E}_g$, it cannot be eliminated in a point by local coordinate of transformations: it demonstrates the presence of a gravitational field by comparing the effect on nearby free falling objects.

ii) The field equation of Newton's theory of gravity, namely Poisson's equation is obtain by imposing that the trace of the tidal tensor is prescribed by ($4\pi G$ times) the density of matter:

$$\Delta\phi = \mathrm{Tr}\left(\frac{\partial^2\phi}{\partial x^i \partial x^j}\right) = 4\pi G\rho \ . \tag{1.47}$$

General relativity will follow the spirit of these two remarks above:

i) Free-falling particles will follow trajectories of vanishing 4−acceleration. Gravity will manifest by the relative acceleration of these non-accelerated trajectories, geometrically encoded in the (Riemann) curvature that represents a tidal field.

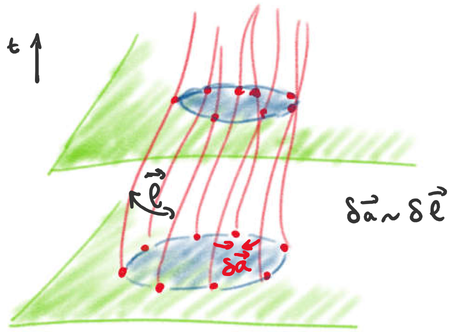

Figure 1.5: Tidal deformations, modifying the trajectories of particles in free fall.

ii) Einstein equations will follow by prescribing the trace of the tidal (curvature) field to be fixed by the appropriate generalization of the mass density in Newtonian dynamics to the appropriate notion in special relativity (namely the so-called stress energy tensor).

Example. Consider the gravitational potential created by spherical distribution of mass M in its exterior

$$\phi = -G\frac{M}{r} \ . \tag{1.48}$$

with $r = \sqrt{x^2 + y^2 + z^2}$. Then, the calculation of the tensor field yields

$$\mathcal{E}_{ij} = -\frac{GM}{r^5}\left(3x_i x_i - \delta_{ij} r^2\right) \ . \tag{1.49}$$

In particular, if we consider a test particle at $\vec{x} = (0, 0, z)$ we find

$$\mathcal{E} = \frac{GM}{r^3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \ , \tag{1.50}$$

so that, writing $\vec{\ell} = (\delta x, \delta y, \delta z)$, we find

$$\begin{aligned} \delta a_x &= -\delta x \mathcal{E}_{xx} = -\delta x \frac{GM}{r^3} \\ \delta a_y &= -\delta y \mathcal{E}_{yy} = -\delta y \frac{GM}{r^3} \\ \delta a_z &= -\delta z \mathcal{E}_{zz} = \delta z \frac{2GM}{r^3} \ . \end{aligned} \tag{1.51}$$

As a consequence, a set of free-falling particles are pull apart in the falling direction and are squeezed in the transversal one. As the tides on Moon and Earth.

1.1.6 What must be retained.

i) Spacetime presents curvature to account for gravity.

ii) Curvature is fixed dynamically, in terms of the presence of mass and energy.

iii) No symmetry imposed to spacetime: absence of (global) inertial frames. Coordinates just *labels* for events in spacetime and have no intrinsic physical meaning. *Inertial observers* are associated with spacetime trajectories in *free fall*, that is, not subject to acceleration.

iv) (Weak) Equivalence principle: locally (at each point) we must recover special relativity, since gravitational forces can be eliminated. Light cones persist, "locally" they are untouched, but their relative distribution is distorted by the spacetime curvature:

- Light travels along *straight* lines along these null cones. It is *deviated* due to spacetime curvature.
- Massive particle travel inside the null cones.
- In the absence of forces additional to gravity, particles follow trajectories of vanishing 4−acceleration, this corresponding to geodesics in the spacetime structure and physicall corresponding to *inertial observers* in *free fall*.
- In particular, light cones can be "forced to turn" in the field of rotating bodies: *frame dragging*.

1.2 Classical collapse: standard relativistic paradigm

As discussed above, a characteristic feature of General Relativity and, more generally of theories modeled on curved spacetimes, is the *bending* of light. Black holes constitute a dramatic extreme in which the light bending is so strong that it cannot leave a certain compact region of the space.

A natural starting point to the study of black holes is to consider the ultimate fate of stars of sufficiently high mass. This *gravitational collapse* approach is not the only possible avenue to the black hole problem, but it has the virtue of providing a general framework that illustrates some of the main aspects, not only of black hole physics, but also of gravitational physics, this including in particular General Relativity. Moreover, it also follows the historical route to the topic.

Let us give a brief overview of the current standard picture of classical gravitational collapse, that constitutes what one might call the *establishment picture of gravitational collapse*. This consists in a heuristic chain of theorems and conjectures providing a general conceptual framework:

i) *Singularity theorems* (Theorem). If enough *energy* is placed in a sufficiently compact region, such that light bending forces the local convergence of all emitted light rays and so-called "trapped surfaces" are formed, then a singularity develops in spacetime [7, 3, 4, 2].

ii) *(Weak) Cosmic Censorship* (Conjecture). In order to keep the predictability of the theory, the formed singularity should be hidden from a distant observer behind a so-called "event horizon", giving rise to a black hole region.

iii) *Spacetime stability* (Conjecture). If general relativity is a physically consistent theory of gravity, it is natural to expect that a system with a finite amount of energy must be eventually driven dynamically to stationarity. This is again a conjecture, now about the stability of a black hole spacetime.

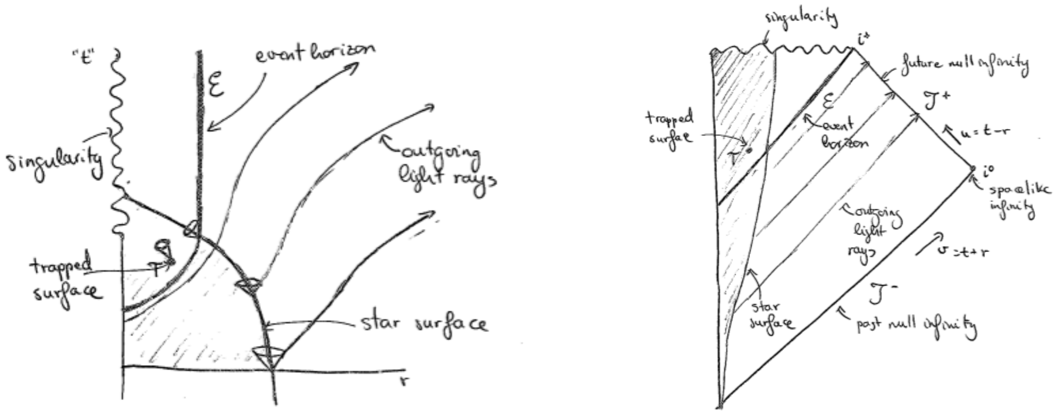

Figure 1.6: Establishment picture of gravitational collapse. The picture in the right is a Carter-Penrose spacetime diagram where lightlike rays lay at $\pm 45^o$. The thick line at 45^o line represents the event horizon, separating the black hole region to its left (containing the spacetime singularity corresponding to the horizontal oscillating line) from the rest of the spacetime.

iv) *Black hole uniqueness* (Theorems). The eventual stationary state is completely characterized by the mass and angular momentum of a the resulting (Kerr) black hole. This is usually referred to as the *no-hair* property of stationary black holes.

The *establishment* picture provides a general systematic framework for posing and addressing issues related to black hole spacetimes. In particular it provides a working program to the study of many of the key aspects to General Relativity. On the other hand, it must be said that nearly every single aspect of it is challenged at one place or another in gravitational physics. In quite a literal sense, the goal of this course is to explain the diagram in Figure 1.2.

1.3 Interest in Black Hole physics

Why should done study black holes? A straightforward valid astrophysical answer could be, simply, because they are out there. Although this is indeed a valid answer, this does not make justice to the richness of the subject. Black holes indeed constitute, on the one hand, crucial ingredients for the understanding of astrophysical and cosmological processes. But, on the other hand, they also provide clues for the understanding of fundamental issues in the theory as well as a cornerstone in modern developments in theoretical physics.

1.3.1 Black holes in astrophysics and Cosmology

Compacity parameter

By now we have a general broad picture of the destiny of star attending to its final mass. The resulting final stage is a compact massive object, namely white dwarf stars, neutron stars or black holes. One might expect that the key parameter controlling the transition from white dwarfs to black hokes to be the density of the final object, but this not quite so. Indeed the (formal) density of supermassive black hole can be indeed very small. The relevant parameter is the one controlling the ability of emitted light rays to escape from the object, and this is controlled by a dimensionless parameter Ξ referred to as the *compacity parameter*

$$\Xi = \frac{GM}{c^2 R} \quad , \tag{1.52}$$

where M is the mass of the object and R is its characteristic scale (radius). In order to gain a qualitative intuition of why the radius enters with as R^{-1}, and not as R^{-3} as it would be the case for a density, it is enough to consider the Newtonian description of the escape velocity. For this we consider a particle of mass m emitted with velocity v from the surface of an spherical object of mass M at radius R. Its total energy is $E_R = \frac{1}{2}mv^2 - \frac{GMm}{R}$. The escape velocity is the one that permits the particle to reach an infinity distance with vanishing velocity, so that $E_\infty = 0$. Conservation of energy then gives

$$\frac{1}{2}mv^2 - \frac{GMm}{R} = 0 \Leftrightarrow \frac{1}{2}v^2 = \frac{GM}{R} \quad . \tag{1.53}$$

Considering the existence of maximum velocity $v = c$, for radius $R < \frac{2GM}{c^2}$ no particle can escape to infinity (this argument was presented already by Michell and Laplace). In other words, for a spherical object if the rate $\frac{GM}{c^2 R}$ is larger than $\frac{1}{2}$ no light can escape. Remarkably, this estimation in Newtonian theory results to be exact when revisited in the context of General Relativity, as we will see in Lecture 5. This justifies the use of (1.52) as the relevant parameter in this context. We provide

Object	M ($M_\odot$)	R (km)	Density (kg/m^3)	Ξ
Earth	3×10^{-6}	6×10^3	5×10^3	10^{-10}
Sun	1	7×10^5	10^3	10^{-6}
White Dwarf	$\sim 0.1 - 1.4$	$\sim 10^4$	10^{10}	$10^{-4} - 10^{-3}$
Neutron Star	$\sim 1 - 3$	~ 10	10^{18}	0.2
Stellar Black Hole (spherical)	$>\sim 3$	$9 (M = 3M_\odot)$	-	0.5
Stellar Black Hole (extremal)	$>\sim 3$	$4.5 (M = 3M_\odot)$	-	1
Massive Black Hole	$\sim 10^9$	$20 U.A.$	-	$0.5 - 1$

Types of black holes

Attending to their mass we can classify black holes in different types:

i) Stellar mass black holes: $M \sim 3 - 30 M_\odot$.

These black holes are predicted by the gravitational collapse description discussed above, starting from highly massive stars. In this sense, they were predicted by the theory.

ii) Massive and supermassive black holes: $M \sim 10^5 - 10^9 M_\odot$.

Black hole of these masses came as a surprise from the need to explain the sources of energy associated with quasars (*quasi-stellar objects*). These are objects at very far distances emitting enormous amounts of energy and finally identified with active galactic nuclei emitting in X-ray, ultraviolet and radio. The emission is around three orders of magnitude that of the total optical luminosity of the parent galaxy. Supermassive black holes at the center of the galaxy offer a mechanism for the generation of such amounts of energy, at the expense of their huge gravitational energy. Along the years the black hole paradigm has become established in the understanding of the properties and evolution's of galaxies.

iii) Intermediate mass black holes: $M \sim 10^3 M_\odot$

There is no unambiguous evidence of the existence of black hole with these masses. They can play an important role in certain astrophysical processes and could be natural intermediate stages between stellar and massive black holes. However there is no observational evidence of their existence.

iv) Primordial black holes: mass up to $\sim 1M_\odot$.

These are hypothetical black holes formed at early stages in the cosmological evolution of the Universe from the collapse of over-dense matter regions. They could play an important role to explain the formation of cosmological structures in the Universe.

Evidence of black holes

i) *Stellar black holes.* Best candidates for stellar black holes are in binaries in which the companion is a normal (non-compact star) providing a flow of material into the black hole. Such material is heated as it forms an accretion disc, emitting in X-rays. From the determination of the orbital parameters one can infer the mass of *dark* object. If the mass is over $3M_\odot$ is a candidate for a black hole and one aims to refine the assessment as a black hole. For this, one can try to identify some of the signatures about the black hole presence provided by general relativity, e.g. i) absence of a rigid boundary surface, existence of an *innermost stable circular orbit* (see Lecture 9) affecting the properties of matter accretion discs, broadening of the $FeK\alpha$ line by gravitational redshift, characteristic distribution of mass and rotation multipoles...

See Table 1.1 in [1] for the best known 22 candidates. These studies, together with evolutionary models and observation of massive stars indicates that stellar black holes are actually very common objects. In our galaxy, the Milky Way, they are estimated to be around $10^8 - 10^9$, something corresponding to a fraction around $10^{-2} - 10^{-3}$ of the total number of stars (around 10^{11} in the Galaxy).

From an astrophysical point of view, stellar mass black holes are important ingredients in the explanation of jet structure of so-called micro-quasars or in models of (long) γ-ray bursts.

ii) *Massive black holes.* Although the mechanism of formation of these black holes is not known, massive and supermassive black holes stand as key ingredients in the most probable explanation of the galactic nuclei activity.

These black holes are at the core of the mechanism for the emission of relativistic jets. They are also able to provoke the tidal disruption of non-compact stars falling onto them and showing a characteristic flares in the electromagnetic spectrum. Maser radiation from quasars also opens a tools to measure parameters of black holes. Finally, it is worthwhile to note that quite recent observations of individual stars of the galactic center of the Milky Way (namely $SgrA*$) have permitted to establish the mass of the black hole at the center of our Galaxy as $4.6 \cdot 10^6 M_\odot$. The tools are very similar to the ones employed in the determination of the mass from the kinematics of binary systems.

Before ending this subsection we note that black holes in general, and binary black holes in particular, stand among the most important sources of gravitational radiation (see Lecture 11). The gravitational radiation emitted from the surrounding of a black hole portrays very characteristic signals of the dynamical spacetime geometry. In this sense, the ultimate tool to identify a compact object as a black hole is provided precisely from the radiation made of the same fabric as black holes: spacetime dynamics.

Black holes as *basic* objects in General Relativity

Black holes are not only relevant because of the role in some of the most violent events in the Universe in astrophysical and cosmological scenarios. They are objects of enormous theoretical interest on their own: on the one hand they represent particularly simple and clean probes into the strong-field regime of general relativity, and on the other hand they stand as a cornerstone piece in the puzzle of bringing together physics at different level of description, namely gravity, quantum mechanics and thermodynamics.

We simply list here some of the relevant aspects of black holes at a theoretical level:

- *Simple classical objects.* Black holes are simple strong gravity solutions in General Relativity. In fact, due to the "no-hair" theorems, in stationarity they are so simple that they can be described only and completely by two parameters. This is extraordinarily singular for a macroscopic object.

- *Two-body problem in general relativity.* Given that general relativity deals essentially with extended objects, the resolution of the motion problem is a very complicated problem by itself, that becomes only more complicate if we add the complexity associated to matter structure. In this sense, black holes provide a particularly clean "equation of state" to study in particular the binary problem in general relativity without having to bother simultaneously with hydrodynamical, rather than gravitational dynamics.

- *Probes into general relativity strong-field regime.* General relativity is well tested in the regime of weak gravitational fields, in particular through the dynamics of binary pulsars. However, the dynamics of the strong field regime and in particular the control and understanding of the decay properties of fields propagating in a strongly dynamical spacetime are poorly understood. Black holes provide a particularly suited probe to study both the stationary and dynamical aspects of the classical gravitational field.

- *Black hole thermodynamics.* The application of general relativity to black hole dynamics leads to a series of laws in perfect analogy with those of thermodynamics. The analogy reached a sounder physical status after the understanding by Hawking that a black hole actually radiate energy according to the black body spectrum of an object in thermal equilibrium, when semiclassical corrections are taking into account. This thermodynamical-like result stands as a solid prediction of the interplay between gravity and quantum mechanics and offers a test for any theory attempting to develop a quantum description of gravity.

- *Cornerstone at the gravity, quantum mechanics and thermodynamical crossroad.* The statistical mechanics understanding of the entropy of a black hole in terms of the number of states of the underlying system, is one of the most important task in approaches to quantum gravity. It offers a test, but also insight to develop avenues into the problem of marrying gravity and quantum mechanics. On the other hand, the evaporation of the black hole through Hawking radiation raises the issue of the unitarity of the black hole evolution description, leading to the black hole *information loss* problem.

- *Black holes in higher dimensions.* Motivated by quantum gravity scenarios involving higher spacetime dimensions (namely *string theory*), there is an interest in understanding classical solutions in higher dimensions presenting an event horizon. First, the uniqueness results associated with the "no hair" property of black hole is lost, offering a more complex panorama. Second, so-called *micro black holes* of up to $\sim 1M_\odot$ appear in speculative theories inspired in so-called *brane worlds*. Third, unexpected mathematical properties shared with four-dimensional black holes are maintained (namely the so-called *hidden-symmetries*), calling for a still missing explanation.

1.4 Summary of Lecture 1

1. Gravitational collapse and mass:
   - i) Compact stars: radius decreases with mass.
   - ii) Maximal mass for white dwarfs and neutron stars.
   - iii) No known mechanism to stop the collapse above $\sim 3M_\odot$.
2. Black holes as a dramatic extreme case of light bending:
   - i) Tension: Special Relativity AND Gravity.
   - ii) Gravitational Redshift: incompatibility with flat spacetime.
   - iii) Spacetime curvature: bending of light.
3. Standard picture of classical gravitational collapse:
   - i) Chain of theorems and conjectures.
   - ii) A conceptual framework for black holes (...and a "Course Program").
   - iii) Every point in the framework is challenged.
4. Interest in Black Holes:
   - i) Astrophysical and Cosmological.
   - ii) Clean probe into the structure of the gravitational theory: General Relativity.
   - iii) A key to physics unification and to new physics.

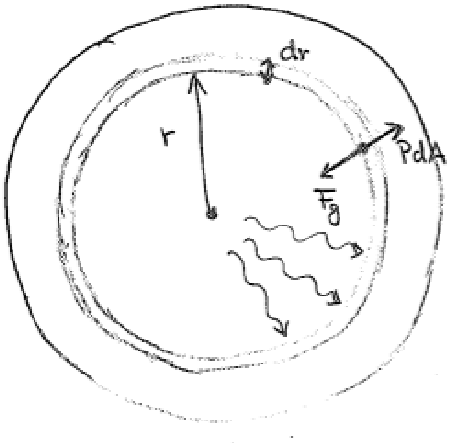

Figure 1.7: Star as a equilibrium between gravitational force and expanding pressure.

1.5 The classical standard picture of gravitational collapse: a first physical overview

1.5.1 Star structure

We start by considering a simplified Newtonian description of stars. The structure of stars is basically governed by three simple laws, namely hydrostatic equilibrium, energy transport and energy generation. For a spherical symmetric star (see Fig. 1.5.1):

$$\begin{aligned}
\frac{dM(r)}{dr} &= 4\pi r^2 \rho(r) \\
\frac{dP(r)}{dr} &= -\frac{GM(r)}{r^2}\rho(r) \quad \text{(hydrostatic equilibrium)} \\
\frac{dL(r)}{dr} &= 4\pi r^2 \epsilon\, \rho(r) \quad \text{(energy conservation)} \\
\frac{dT(r)}{dr} &= -\frac{1}{4\pi r^2 \lambda} L(r) \quad \text{(energy transport)}
\end{aligned}$$

where the primary variables of the system $M(r), P(r), L(r), T(r)$:

$M(r)$: mass contained from the center $r = 0$ to the shell of radius r

$P(r)$: pressure at radius r

$L(r)$: energy flow through the sphere of radius r

$T(r)$: temperature at radius r.

In order to close the system we need:

- Equation of state: $P = P(\rho, T, X_i)$, or inverting $\rho = \rho(P, T, X_i)$
- Coefficient of conductivity: $\lambda = \lambda(\rho, T, X_i)$
- Energy production rate: $\epsilon = \epsilon(\rho, T, X_i)$

with X_i accounting for the chemical composition. In addition we need boundary conditions. This would parametrize the stars in terms of its radius. However, the radius is a bad parameter since it is difficult to determine either experimentally or *a priori*. A better choice is to choose the mass of the star. For this we rewrite (1.54), with the mass contained inside a given shell as parameter:

$$\begin{aligned}
\frac{dr(M)}{dM} &= \frac{1}{4\pi r^2 \rho(M)} && \\
\frac{dP(M)}{dM} &= -\frac{GM}{4\pi r^4} && \text{(hydrostatic equilibrium)} \\
\frac{dL(M)}{dM} &= \epsilon && \text{(energy conservation)} \\
\frac{dT(M)}{dM} &= -\frac{1}{16\pi^2 r^4 \lambda \rho(M)} L(M) && \text{(energy transport)}
\end{aligned}$$

Appropriate (approximate) boundary conditions are:

$$r(0) = 0 \ , \ L(0) = 0 \ , \ P(M_{\rm star}) = 0 \ , \ T(M_{\rm star}) = 0$$

where $M_{\rm star}$ is the total mass of the star, which becomes a parameter in the model.

The crucial ingredients to counteract the gravity and keep hydrostatic equilibrium are the energy production rate and the equation of state. In gravitational collapse, part of the initial gravitational energy is used to heat the matter. However, the resulting increase in the pressure is not enough to reach the hydrostatic equilibrium. When the temperature is high enough nuclear reactions are initiated and the resulting ϵ is able to keep the equilibrium and the life of star is span. However, once this nuclear fuel is exhausted, the hydrostatic equilibrium is once more lost and collapse continues. The collapse continues until matter reaches an stage in which the equation of state is *rigid enough*. This leads to the formation of compact stars.

1.5.2 Compact stars

Degenerate Fermi gas. Fermions satisfy Pauli's exclusion principle, that prevents two fermionic particles to be in the same quantum state. Electrons, protons and neutrons are fermionic particle of spin 1/2. This in particular means that for a given momentum p there can only be two particles (spin-up and spin-down). As a consequence, particles occupy the phase space till a maximum Fermi momentum $p_{\rm F}$. As a consequence of this motion, the resulting *degenerate Fermi gas* acquires a pressure. It is this pressure that balances the gravitational force.

In our context the relevant particles are electrons and neutrons since, at sufficiently high densities, protons and electrons suffer a weak force process (a form of beta-decay) known as *neutronization*:

$$e^- + p^+ \to n^0 + \nu_e \tag{1.54}$$

The equation of state of a degenerate Fermi gas has two different regimes: i) non-relativistic regime, when the reached Fermi momentum satisfy $p_{\rm F} \ll mc$ and ii) the ultra-relativistic regime, when $p_{\rm F} \gg mc$. The equations of state differ in both cases, although they share the key feature of not depending on the temperature. We have (see e.g. [1])

$$\begin{aligned}
&\text{relativistic Fermi gas:} && P = K\frac{\hbar^2}{m}(\tfrac{N}{V})^{\frac{5}{3}} \\
&\text{ultra-relativistic Fermi gas:} && P = K'\hbar c(\tfrac{N}{V})^{\frac{4}{3}}
\end{aligned} \tag{1.55}$$

where N is the total number of fermions and K and K' are dimensionless constants.

Degenerate stars. Estimating the density as $\rho \sim M/R^3$ and the pressure gradient as $\nabla P \sim P/R$ we can write the hydrostatic equilibrium equation as $\frac{GM\rho}{R^2} \sim \nabla P$

$$GM^2 \sim PR^4 \tag{1.56}$$

We also introduce the mass per Fermi particle $m' = M/N$. Then, we can write:

- *Non-relativistic regime*: From Eq. (1.55)

$$P \sim \frac{\hbar^2}{m}\left(\frac{N}{V}\right)^{\frac{5}{3}} \sim \frac{\hbar^2}{m} \cdot \frac{N^{\frac{5}{3}}}{R^5} \tag{1.57}$$

  so that from (1.56) we have

$$GM^2 \sim \frac{\hbar^2}{m} \cdot \frac{N^{\frac{5}{3}}}{R} \tag{1.58}$$

  and using m'

$$R \sim \frac{\hbar^2}{Gmm'^{5/3}} \frac{1}{M^{1/3}} \tag{1.59}$$

  From this we conclude that the larger the mass, the smaller the radius. This is the crucial ingredient of the Fermi degenerate equation of state. It implies that as we consider increasing masses the density and pressure also grow until we reach a (ultra-)relativistic regime for the Fermi gas.

- *Ultra-relativistic regime*: Repeating the steps:

$$P \sim \hbar c \left(\frac{N}{V}\right)^{\frac{4}{3}} \sim \hbar c \cdot \frac{N^{\frac{4}{3}}}{R^4} \tag{1.60}$$

  and

$$GM^2 \sim \hbar c \cdot N^{\frac{4}{3}} \tag{1.61}$$

  Remarkably, the radius disappears from the equilibrium relation, so that the mass is fixed

$$M \sim M_\star = \frac{(\hbar c/G)^{3/2}}{m'^2} \tag{1.62}$$

The conclusion is that for masses below $M_\star$, the pressure associated with the degenerate Fermi gas supports the gravitational force. As the mass increases the radius decreases and the fermions become more and more relativistic. Then the ultra-relativistic regime provides the critical mass mass that can be supported by this mechanism.

White dwarfs are compact stars in which the degenerate Fermi gas is composed of electrons. In this case, the limit to the mass is known as the *Chandrasekhar limit* and is about $1.44 M_\odot$. For neutron stars, resulting from supernova core-collapses of massive stars, the limit is referred to as *Tolman-Oppenheimer-Volkoff* and is less precisely established, depending essentially on the details of the equation of state. A particular (exotic) class of *neutron* star are *quark stars* in which the relevant degenerate fermions are *strange stars* (postulated as the ground state of baryonic matter).

Beyond this mass, no mechanism is known capable of stopping the gravitational collapse. The eventual result of this process is what we know as *black hole*. Black holes are a dramatic extreme case of a characteristic feature of General Relativity: *bending of light*. And the latter is a manifestation of a more general concept: spacetime curvature. Let us explore how this concept emerges in the study of gravitation.

Chapter 2

Gravity as spacetime curvature I: manifolds, vector fields, spacetime metric

Contents

2.1 Events in *spacetime*: manifolds and coordinates

Notion of manifold to describe events in space and time. Absence of *a priori* given structures in General Relativity: all objects are fixed dynamically. Coordinates understood as *labels* without intrinsic meaning: need of coordinate independence of physical statements.

2.1.1 The manifold of physical events

Newtonian description.

Let us start by considering the description of a point-like physical process happenning in space and time in the context of Newtonian physics. A fundamental tenet in the theory is the existence of a special class of reference frames in which Newton laws apply: these are called *inertial frames*. This provides an "a priori" structure in the theory, "rigid" in the sense that does not result from any dynamical equations. In particular, such frames provide a set of spatial coordinates (x, y, z) and a time coordinate t, permitting to associate a "time label" and a "space label" with any physical "event" p, say the presence of a particle:

$$p \mapsto (t_p, x_p, y_p, z_p) \tag{2.1}$$

Considering a physical point-like particle to fix ideas, its evolution in time and space is described by a "trajectory" parametrized by a label λ:

$$p(\lambda) \mapsto (t_p(\lambda), x_p(\lambda), y_p(\lambda), z_p(\lambda)) \tag{2.2}$$

In Newtonian physics, t is a universal parameter and it is natural to use $\lambda = t$ so that $(x_p(t), y_p(t), z_p(t))$ describe the trajectory. When writing down the dynamical equations, we have the freedom to choose (x, y, z) up to a Galilean tranformation: translations, rotations and boosts.

$$t' = t + t_0 \ , \ \vec{x}' = \vec{x} - \vec{a} \ , \ \vec{x}' = R(\epsilon) \cdot \vec{x} \ , \ \vec{x}' = \vec{x} - \vec{v}t \quad , \quad R(\vec{\epsilon}) \in SO(3) \tag{2.3}$$

Here $\vec{\epsilon}$ can be parametrised, say, by the Euler angles. As an example of a rotation, we make explicit the rotation of angle φ (Euler angle α) around the z axis, *mixing* the x and y coordinates

$$R(\varphi, 0, 0) = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & & 1 \end{pmatrix} \tag{2.4}$$

On the other hand, coordinates (x, y, z) have a geometric content as associated with inertial frames. In particular rotations preserves the Euclidean metric in $\mathbb{R}^3$: namely the symmetric, definite-positive, non-degenerated quadratic form $\boldsymbol{\delta} = \text{diag}(1, 1, 1)$

$$R\boldsymbol{\delta}R^t = \boldsymbol{\delta} \quad \Leftrightarrow \quad RR^t = \mathrm{I} \ , \tag{2.5}$$

forming the $SO(3)$ when we require orientation preservation.

Special relativity.

The same reasoning essentially applies to special relativity. Although time is no longer absolute, the notion of inertial frame exists, providing an a priori structure for the description of physical events. The freedom in the choice of $\mathbf{x} = (ct, x, y, z)$ is up to a Poincaré transformation (namely affine transformations preserving the symmetric, non-degenerated, signature $(1, 3)$, quadratic form $\boldsymbol{\eta} = \text{diag}(-1, 1, 1, 1)$), where time and spatial coordinates are "mixed"

$$\mathbf{x}' = \mathbf{a} + \Lambda \cdot \mathbf{x} \quad . \tag{2.6}$$

In particular linear transformations associated with matrices Λ preserve the Minkowski metric $\text{diag}(-1, 1, 1, 1)$ in $\mathbb{R}^4$

$$\Lambda\boldsymbol{\eta}\Lambda^t = \boldsymbol{\eta} \ , \tag{2.7}$$

spanning the Lorentz group $SO(1, 3)$. We make explicit the form of a boost along the x direction with velocity v

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\frac{v}{c}\gamma & 0 & 0 \\ -\frac{v}{c}\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \tag{2.8}$$

with $\gamma = \dfrac{1}{\sqrt{1-(v/c)^2}}$ (the so-called Lorentz factor, needed to guarantee the unity of the squared determinant of Λ, following from the preservation of the Lorentz metric). Noting $\gamma^2 - (v/c)^2\gamma^2 = 1$, we can write the boost matrix as

$$\begin{pmatrix} \cosh\alpha & -\sinh\alpha & 0 & 0 \\ -\sinh\alpha & \cosh\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.9}$$

where $\tanh\alpha = -v/c$. Note that, as the rotation R in (2.4) mixed coordinates x a y, the boost trasnformation in (2.9) acts as a kind of *rotation* in the (t, x) subspace. In sum, also in special relativity a geometric meaning is associated with the coordinate structure of inertial frames.

General Relativity.

A basic tenet in the general relativistic description is that all structures in the theory must be determined dynamically, through the resolution of the appropriate equations. In particular, this means that the a priori notion of (global) inertial reference frame is absent. Still, in order to have an analytical description, we need to associate to a physical event p some "labels" (t, x, y, z), as in (2.1). However, now the "coordinates" (t, x, y, z) are completely devoid of geometric or physical meaning. They are simply labels without intrinsic meaning and the dynamical description should be independent of them.

Physical statements must be also independent of the choice of coordinates. As an example, the coordinate description of an object trajectory has no intrinsic physical meaning. Different descriptions are possible, none of them being privileged. Physical statements become *relational statements*. For instance, the notion of "position" of a particle has no intrinsic meaning by itself, but the "crossing" of two particles does. That is, the meeting of two objects along their dynamical evolutions has an intrinsic physical meaning: the fact that the two trajectories cross is independent of their coordinate description (cf. Fig. 2.1.1). Such "meeting" provides an example of a spacetime "event", that we model as a "point" in an appropriate space.

Spacetime manifold.

Spacetime is the ensemble M of all physical(/geometric) intrinsic events. As such, M is an abstract space. We require some structure on this space, in particular an appropriate topology providing with some basic notions of continuity that we would like to promote to our modelling of this set of physical "events" [1].

[1] In particular, in our modelling of spacetime events we would like to be able to tell events apart, namely to be able to refer to different event as "separate" points, in such a way that for any two points in M there should exist respective neighbourhoods of each of them which are disjoint. This is captured by the notion of "Hausdorff" space (also "separated" or T_2 space). A second (technical) requirement to promote "local proofs" to a global stage is that there should be exist a countable collection $\mathcal{U}$ of open sets, such that any open set in the topology can be written as the union of a family of open sets in $\mathcal{U}$. This chracterizes M as a "second-countable" space. As referred above, this is a technical requirement that is key to use tools such the "partition of unity" to "glue" local results into global proofs. The most important requirement is however that the space M should be locally homeomorphic to $\mathbb{R}^n$, for some $n < \infty$. We refer to [6] for an appropriate presentation of these topological notions in the present context.

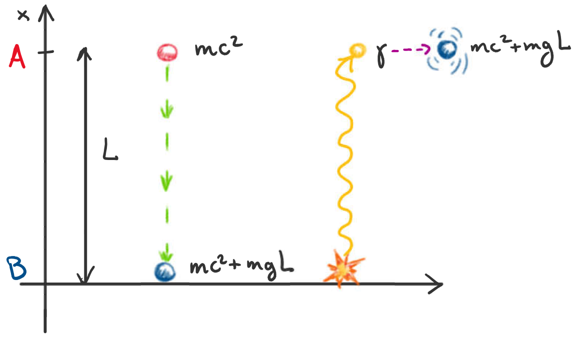

Figure 2.1: Two "observers" γ_1 and γ_2 travelling along their respective trajectories meet at points p_1 and p_2. These two points constitute intrinsic "spacetime events", that can be represented with different sets of coordinate labels.

The most important of these requirements is that we require that spacetime events can be locally parametrized by formal time and space labels (the sense in which they refer to "time" and "space" notions will be discussed later, once the notion of metric is introduced). That is, independently of its global structure, M should locally look as $\mathbb{R}^n$: we require that M can be locally patched to open sets in $\mathbb{R}^4$.

More specifically, this leads to the notion of *local chart*, that is simply a way of parametrizing an open set $U \subset M$ by an open set $\tilde{U} \subset \mathbb{R}^n$:

$$\begin{aligned} \varphi : U \subset M &\rightarrow \tilde{U} \subset \mathbb{R}^n \\ p &\mapsto x^\mu \equiv (x^0, \ldots, x^{n-1}) \end{aligned} \tag{2.10}$$

We require that φ is an homeomorphism (continuous with continuous inverse), so that at the local level the topology of M is that of $\mathbb{R}^n$. We do not have "access" directly to p, but to its coordinate representation $x^\mu = (x^0, \ldots, x^{n-1})$. The coordinate representation has no physical/geometrical content, and different labelings are possible:

$$\begin{aligned} \varphi_1 : U_1 \subset M &\rightarrow \tilde{U}_1 \in \mathbb{R}^n \quad , \quad & \varphi_2 : U_2 \subset M &\rightarrow \tilde{U}_2 \in \mathbb{R}^n \\ p &\mapsto (x^0, \ldots, x^{n-1}) \quad , & p &\mapsto (y^0, \ldots, y^{n-1}) \end{aligned} \tag{2.11}$$

so that

$$\begin{aligned} \varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \subset \tilde{U}_1 \subset \mathbb{R}^n &\rightarrow \varphi_2(U_1 \cap U_2) \subset \tilde{U}_2 \in \mathbb{R}^n \\ (x^0, \ldots, x^{n-1}) &\mapsto (y^0, \ldots, y^{n-1}) \end{aligned} \tag{2.12}$$

is an homeomorphism between open sets[2] in $\mathbb{R}^n$. In simple terms this represents a change of coordinates in the local description of M in $U_1 \cap U_2$:

$$y^i = y^i(x^0, \ldots, x^{n-1}) \quad , \quad i \in \{1, \ldots, n-1\} \, . \tag{2.13}$$

The spacetime st M is covered by a collection of charts (U_i, φ_i) and their transition functions $\varphi_{ij} = \varphi_j \circ \varphi_i^{-1}$. Such complete collection of charts is called an *atlas* of M. This confers M

[2]Note that the domain and codomain of the "change of coordinates" $\varphi_2 \circ \varphi_1^{-1}$ is actually determined by the restriction in which both coordinate charts are valid, that explaining the given expression.

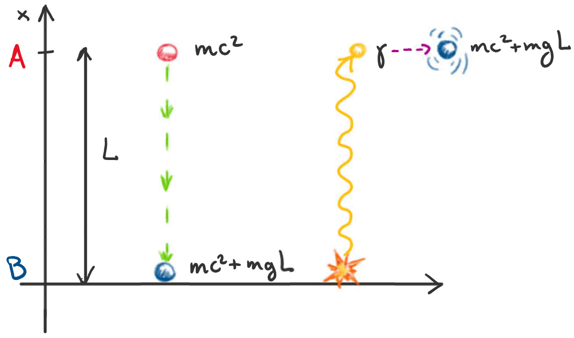

Figure 2.2: Two "local charts" $(U_1, \varphi_1 : U_1 \to \tilde{U_1})$ and $(U_2, \varphi_2 : U_2 \to \tilde{U_2})$ covering the crossing of the two trajectories γ_1 and γ_2. They provide two local representations of the same underlying geometric objects. We access points in $\tilde{U_1}$ and $\tilde{U_2}$, never in M. The key element for the reconstruction of M from charts are the set of "transition functions" $\varphi_{ij} = \varphi_j \circ \varphi_i^{-1}$.

with the structure of a (topological) *manifold*. To further qualify the manifold M, we need to impose further structure. The specific type of manifold we work with depends on the properties we enforce on the transtion functions φ_{ij}. Here, we will require functions $\varphi_{ij} = \varphi_j \circ \varphi_i^{-1}$ to be C^∞ diffeomorphisms between charts $\tilde{U}_i$ and $\tilde{U}_j$. In other words, we are going to work with infinite-differentiable changes of coordinates. This is a practical choice, but nothing guarantees that it encodes the actual regularity of spacetime: this is not jst an academical issue, it actually determines what we can prove and what we can, as we will see when discussing black hole uniqueness theorems [3].

2.2 Vectors and one-forms at a point. Tangent and cotangen spaces

Vectors on a manifold: tangent space. Vectors as derivations. Contravariant and covariant tensors. Tensors as local (point-like) objects on the manifold. Passive and active views of coordinates changes (diffeomorphisms): push-forward and pull-back transformations. Lie derivative as tensor variation along an infinitesimal diffeomorphism.

2.2.1 Linear approximation of the spacetime: tangent plane

We need more structure in order to manipulate efficiently the geometrical/physical objects. In particular, we aim at introducing the structure needed to translate the objects from calculus. This leads us to consider the approximation of the manifold M by linear structures.

figure derivative

This is in the very same spirit of approximating a non-linear differentiable function $f : \mathbb{R} \to \mathbb{R}, x \mapsto f(x)$ at a point x by its derivative

$$f(x+h) = f(x) + h\frac{df}{dx}(x) + o(h) \ , \tag{2.14}$$

[3] A challenging question in this setting is: *what is the actual (fundamental or effective) "regularity" of spacetime?*

or, more generally, a non-linear differentiable application between open sets of two linear spaces $f : U \subset \mathbb{R}^n \to V \subset \mathbb{R}^m$ by its differential at x (actually characterized as its best linear approximation at that point x)

$$f(x+h) = f(x) + df(x) \cdot h + o(h) \quad , \quad h \in \mathbb{R}^n \; . \tag{2.15}$$

In this sense, we want to be able to approximate M "close" to a given point p by a *tangent plane* T_pM. This tangent plane provides a linear approximation to M.

2.2.2 Vectors as curve derivatives.

A first way to look at the construction of such linear space T_pM, is to consider vectors in T_pM as derivatives ("velocities") of curves passing through p, i.e. $\gamma : I \subset \mathbb{R} \to M$, with $\gamma(0) = p \in M$. Considering a coordinate description[4] of γ in coordinates $\{x^\mu\}$ associated with a chart (φ_1, U), we can write

$$\begin{array}{rcl} \varphi_1 \circ \gamma : I \subset \mathbb{R} & \to & \tilde{U}_1 = \varphi_1(U_1) \in \mathbb{R}^n \\ \lambda & \mapsto & x^\mu(\lambda) = (x^0(\lambda), \ldots, x^{n-1}(\lambda)) \; . \end{array} \tag{2.16}$$

Note that $\varphi_1 \circ \gamma$ is an application from $I \in \mathbb{R}$ to an open set of $\mathbb{R}^n$. We demand such an application to be differentiable at $\lambda = 0$ (that is, "at" the point $p \in M$). We introduce the components of a vector v in T_pM, in a basis associated with the local chart $\{x^\mu\}$, as

$$v^\mu = \left.\frac{dx^\mu}{d\lambda}\right|_{\lambda=0} = \left.\begin{pmatrix} \frac{dx^0}{d\lambda} \\ \vdots \\ \frac{dx^{n-1}}{d\lambda} \end{pmatrix}\right|_{\lambda=0} \tag{2.17}$$

More precisely, this is the vector in $\mathbb{R}^n$ obtained from the application of the differential $d(\varphi_1 \circ \gamma)(\lambda)$, evaluated at $\lambda = 0$, to the vector $1 \in \mathbb{R}$: $v = d(\varphi_1 \circ \gamma)(0) \cdot 1$. The linear structure in $\mathbb{R}^n$ permits to construct curves passing through $\varphi_1(p) \in \tilde{U}_1$, whose tangent vectors can be additioned and can be multiplied by scalars (this corresponds to the reparametrization of the curve γ). Such curves can be transported to $U \subset M$ through the homeomorphism φ_1^{-1} so that addition and multiplication by scalars can be associated with the coordinate representation of derivatives of curves in M.

In order to make this construction independent of the choice of coordinates, we consider the representation of the derivative of γ at p in another local chart, say

$$\begin{array}{rcl} \varphi_2 \circ \gamma : I \subset \mathbb{R} & \to & \tilde{U}_2 = \varphi_2(U_2) \in \mathbb{R}^n \\ \lambda & \mapsto & y^\mu(\lambda) = (y^0(\lambda), \ldots, y^{n-1}(\lambda)) \; . \end{array} \tag{2.18}$$

To ensure the compatibility of linear structures induced on derivatives of γ at p from both coordinate representations, we impose $\varphi_2 \circ \varphi_1^{-1}$ and its inverse $\varphi_1 \circ \varphi_2^{-1}$ to be differentiable, namely $\varphi_1 \circ \varphi_2^{-1}$ to be a diffeomorphism. In simple terms, the coordinate change (2.13) and

[4] An intrinsic definition of a vector at a point p, not referring to particular coordinates, can be done in terms of classes of equivalence of curves passing through that point p and having the same velocity at it.

its inverse must be differentiable. Indeed, rewriting the same curve γ in the coordinates $\{y^\mu\}$ associated to the chart (φ_2, U_2), and using (2.13) we find[5], using the chain rule

$$\left.\frac{dy^\mu}{d\lambda}\right|_{\lambda=0} = \left.\left(\frac{\partial y^\mu}{\partial x^\nu}\right)\right|_{x(\lambda=0)} \left.\frac{dx^\nu}{d\lambda}\right|_{\lambda=0} , \tag{2.19}$$

where the Jacobian matrix $\left(\frac{\partial y^\mu}{\partial x^\nu}\right)$ is the matrix representation of the differential $d(\varphi_2 \circ \varphi_1^{-1})$

$$\left(\frac{\partial y^\mu}{\partial x^\nu}\right) = \begin{pmatrix} \frac{\partial y^0}{\partial x^0} & \cdots & \frac{\partial y^0}{\partial x^{n-1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial y^{n-1}}{\partial x^0} & \cdots & \frac{\partial y^{n-1}}{\partial x^{n-1}} \end{pmatrix} \tag{2.20}$$

in the canonical bases in the respective linear spaces $\mathbb{R}^n$ corresponding to coordinates $\{x^\mu\}$ and $\{y^\mu\}$. The linearity of the differential guarantees the compatibility of linear structures induced from both charts.

A vector $\boldsymbol{v} \in T_pM$ can then be seen as the quotient of the set of coordinate representations $\{v_x^\mu, v_y^\mu, \ldots, v_i^\mu, \ldots\}$ (in the local charts U_i of the atlas such that $p \in U_i$) by the equivalence class provided by (2.19), namely $v_y^\mu \sim v_x^\mu$ if there exists a differentiable change of variables $y = y(x)$ such that

$$v_y^\mu = \left.\left(\frac{\partial y^\mu}{\partial x^\nu}\right)\right|_{\varphi_1(p)} v_x^\nu . \tag{2.21}$$

The linearity of operations on $\boldsymbol{v} \in T_pM$ defines T_pM as a vector space that we refer to as the *tangent space* to M at p.

As we see, this construction requires the differentiability of the changes of charts $\varphi_i \circ \varphi_j^{-1}$ between any elements of the atlas U_j and U_i (if not, we cannot establish an equivalence relation). This defines a *differentiable manifold* [6].

2.2.3 Vectors as derivations: directional derivatives of a function.

A useful caracterization of the vectors of the tangent space T_pM is given by a generalization of the notion of directional derivative of a function. More specifically, let us consider a function $f : M \to \mathbb{R}$ and a vector $\boldsymbol{v} \in T_pM$. We would like to give a meaning to the directional derivative of f at p along the direction $\boldsymbol{v}$. Let us first consider a curve $\gamma(\lambda)$ in M such that $\gamma(0) = p$ and $d\gamma/d\lambda|_0 = \boldsymbol{v}$. Then, we consider the coordinate representation of a function $f : M \to \mathbb{R}$ in coordinates $\{x^\mu\}$ (in a neighbourhood of p)

$$\begin{array}{rcl} f : \tilde{U}_1 & \to & \mathbb{R} \\ x^\mu & \mapsto & f(x^\mu) = f(x^0, \ldots, x^{n-1}) . \end{array} \tag{2.22}$$

In this local representation we can write the curve as $x^\mu = x^\mu(\gamma(\lambda)) = x^\mu(\lambda)$ (note the slight abuse of notation) so that $x_o^\mu = x^\mu(p) = x^\mu(\lambda = 0)$ and $v^\mu = \left.\dfrac{dx^\mu}{d\lambda}\right|_{\lambda=0}$. We can then calculate

[5] This is the first encounter to the so-called index convention of summation of repeated indices.

[6] Other conditions on the changes between local charts give rise to other type of manifolds, e.g. analyticity, C^k conditions. For simplicity, we will restrain ourselves to C^∞-differentiable manifolds.

the derivative "at p" of the associated local represention of $f \circ \gamma$, that is

$$\left.\frac{df}{d\lambda}\right|_{\lambda=0} = \left.\frac{\partial f}{\partial x^\mu}\right|_{x(\lambda=0)} \left.\frac{dx^\mu}{d\lambda}\right|_{\lambda=0} = \left.\frac{\partial f}{\partial x^\mu}\right|_{x_o} v^\mu = \left(v^\mu \left.\frac{\partial}{\partial x^\mu}\right|_{x_o}\right) f = \boldsymbol{v}(f) \ . \tag{2.23}$$

where we have denoted the vector $\boldsymbol{v}$ as $\boldsymbol{v} = v^\mu \left.\frac{\partial}{\partial x^\mu}\right|_{x_o}$. Relaxing a bit the notation to emphasize the geometric content[7], we can write $\boldsymbol{v} = v^\mu \left.\frac{\partial}{\partial x^\mu}\right|_p$, so that $\boldsymbol{v}$ that can be understood as a derivation on functions at point p

$$\boldsymbol{v}(f) = v^\mu \left.\frac{\partial f}{\partial x^\mu}\right|_p \ . \tag{2.25}$$

This approach to tangent vectors provides a natural notation for the linear basis $\{\boldsymbol{e}^x_\mu\}$ at T_pM associated to $\{x^\mu\}$, as derivations along the coordinates x^μ, namely

$$\boldsymbol{e}^x_\mu \equiv \left.\frac{\partial}{\partial x^\mu}\right|_p \tag{2.26}$$

When there is no possible confusion in the coordinate basis, we will denote $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$. Therefore we can write the vector $\boldsymbol{v}$ as

$$\boldsymbol{v} = v^\mu \partial_\mu|_p \tag{2.27}$$

2.2.4 Dual to the tangent space T_pM: cotangent space T^*_pM.

The dual space T^*_pM to T_pM is the set of linear applications

$$\boldsymbol{\omega} : T_pM \to \mathbb{R} \tag{2.28}$$

Given that T_pM is a linear space of finite dimensions, by duality (namely $\boldsymbol{v}(\boldsymbol{\omega}) = \boldsymbol{\omega}(\boldsymbol{v})$) vectors in T_pM can be seen as linear applications

$$\boldsymbol{v} : T^*_pM \to \mathbb{R} \ . \tag{2.29}$$

Given the (ordered) basis $\{\boldsymbol{e}^x_\mu\}$ in T_pM associated with a coordinate system $\{x^\mu\}$, we can introduce the dual basis in T^*_pM, $\{\boldsymbol{\omega}^\mu_x\}$, that is characterised by

$$\boldsymbol{\omega}^\mu_x(\boldsymbol{e}^x_\nu) = \boldsymbol{e}^x_\nu(\boldsymbol{\omega}^\mu_x) = \delta^\mu{}_\nu \ . \tag{2.30}$$

[7]The notation around Eq. (2.23), taking into account all the composed applications, results a bit cumbersome and may hidden the actual geometric content. In this geometric spirit, we can abuse the notation and identify p with either $\lambda = 0$ or $x^\mu(p) = x^\mu(\lambda = 0)$ through the respective appropriate mappings (namely γ and the local chart $\{x^\mu\}$), so that we can write $v^\mu = dx^\mu/d\lambda|_p$ and

$$\left.\frac{df}{d\lambda}\right|_p = \left.\frac{\partial f}{\partial x^\mu}\right|_p \left.\frac{dx^\mu}{d\lambda}\right|_p = \left.\frac{\partial f}{\partial x^\mu}\right|_p v^\mu = \left(v^\mu \left.\frac{\partial}{\partial x^\mu}\right|_p\right) f = \boldsymbol{v}(f) \ , \tag{2.24}$$

with $\boldsymbol{v} = v^\mu \left.\frac{\partial}{\partial x^\mu}\right|_p$. In the following we will follow the same criterium, unless otherwise specified.

A geometric understanding of ω_x^μ comes naturally in terms of the differential of a function. Let us consider the differential df at p in the local representation of f, as an application $f : \tilde{U}_1 \subset \mathbb{R}^n \to \mathbb{R}$. Because of its linearity, the application $df(p) : \mathbb{R}^n \sim T_pM \to \mathbb{R}$ is an element in the dual T_p^*M. In particular $df(p)$ takes (linearly) vectors $\boldsymbol{v}$ in T_pM to vectors (values) in $\mathbb{R}$ so that, in the chosen local coordinates

$$df(p)\,(\boldsymbol{v}) = \left.\frac{\partial f}{\partial x^\mu}\right|_p v^\mu = v^\mu \left.\frac{\partial f}{\partial x^\mu}\right|_p = \boldsymbol{v}(f)\ , \tag{2.31}$$

where the first step is just the matricial expression of the action of the differential and in the last step we have used (2.25). In particular, by duality $df(p)(\boldsymbol{v}) = \boldsymbol{v}(df(p))$, we have the identities

$$\boldsymbol{v}(df(p)) = df(p)(\boldsymbol{v}) = \boldsymbol{v}(f)\ . \tag{2.32}$$

If we consider now as f the functions $x^\mu : U_1 \to \mathbb{R}$ provided by a local chart, we can evaluate the action of their differentials $dx^\mu(p)$ at p on the elements in the coordinate vector basis $\{\boldsymbol{e}_\mu^x\}$

$$dx^\mu(p)(\boldsymbol{e}_\nu) = dx^\mu(p)\left(\left.\frac{\partial}{\partial x^\nu}\right|_p\right) = \left.\frac{\partial}{\partial x^\nu}\right|_p (x^\mu) = \delta^\mu{}_\nu \tag{2.33}$$

From the characterization (2.30) of the dual basis $\{\boldsymbol{\omega}_x^\mu\}$ we conclude

$$\boldsymbol{\omega}_x^\mu = dx^\mu(p) \equiv dx^\mu|_p\ , \tag{2.34}$$

so that the differentials of the coordinate functions at p provide a basis for T_p^*M, dual to the one in (2.34) for T_pM, given in terms of the partial derivatives. Finally, we can express the linear form $df(p) \in T_p^*M$ in this basis $\{\boldsymbol{\omega}_x^\mu\}$, by calculating its components as

$$df(p)(\boldsymbol{e}_\mu) = df(p)\left(\left.\frac{\partial}{\partial x^\nu}\right|_p\right) = \left.\frac{\partial f}{\partial x^\nu}\right|_p\ , \tag{2.35}$$

so that we can write

$$df(p) = \left.\frac{\partial f}{\partial x^\nu}\right|_p dx^\mu|_p\ , \tag{2.36}$$

that gives a geometric view on the differential of a function f at a point p.

2.3 Vector fields and differential one-forms

Hitherto we have considered vectors tangent to M at a given point $p \in M$, namely in the tangent space T_pM. We are going to consider now the smooth assignment of a vector $\boldsymbol{v}_p$ to each point at $p \in M$. This is the idea of a smooth *vector field* on M, as a "smooth" mapping

$$\boldsymbol{v} : p \in M \quad \mapsto \quad \boldsymbol{v}_p \in T_pM \tag{2.37}$$

In the same spirit, a smooth 1−form on M is a "smooth" mapping

$$\alpha : p \in M \quad \mapsto \quad \alpha_p \in T_p^*M\ . \tag{2.38}$$

Let us denote the ensemble of vector fields on M as $\mathcal{T}M$ and the ensemble of 1-forms as $\mathcal{T}^*M$. Then, we can see a vector field as a $C^\infty(M)$-linear application

$$\begin{aligned} \boldsymbol{v} : \mathcal{T}^*M &\rightarrow C^\infty(M) \\ \boldsymbol{\alpha} &\mapsto \boldsymbol{v}(\boldsymbol{\alpha}) \ , \end{aligned} \tag{2.39}$$

that is smooth in the sense that $\boldsymbol{v}(\boldsymbol{\alpha})$ is a $C^\infty(M)$ real function of M

$$\begin{aligned} \boldsymbol{v}(\boldsymbol{\alpha}) : M &\rightarrow \mathbb{R} \\ p &\mapsto \boldsymbol{v}_p(\boldsymbol{\alpha}_p) \ . \end{aligned} \tag{2.40}$$

That is[8]

$$\boldsymbol{v}(f\boldsymbol{\alpha} + g\boldsymbol{\beta}) = f\boldsymbol{v}(\boldsymbol{\alpha}) + g\boldsymbol{v}(\boldsymbol{\beta}) \quad , \forall f, g \in C^\infty(M) \ , \ \forall \boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathcal{T}^*M \tag{2.41}$$

and $\boldsymbol{v}(\boldsymbol{\alpha}), \boldsymbol{v}(\boldsymbol{\beta}) \in C^\infty(M)$. In analogous manner, a 1-form can be seen as a smooth $C^\infty(M)$-linear mapping

$$\begin{aligned} \boldsymbol{\alpha} : \mathcal{T}M &\rightarrow C^\infty(M) \\ \boldsymbol{v} &\mapsto \boldsymbol{\alpha}(\boldsymbol{v}) \ . \end{aligned} \tag{2.42}$$

Given a local chart with coordinates $\{x^\mu\}$, the bases (2.26) and (2.34) or T_pM and T_p^*M extend to the bases of $\mathcal{T}M$ and $\mathcal{T}^*M$ (as C^∞-modules)

$$\boldsymbol{e}_\mu = \partial_\mu \quad , \quad \boldsymbol{\omega}^\mu = dx^\mu \ . \tag{2.43}$$

2.3.1 Vector fields as function derivations.

We provide an altenative characterization of vector fields and make contact again with the original introduction of a vector at a point p as a derivation of a function f at p. Letting p move in M and assigning vectors smoothly as p changes characterizes also a vector field. Namely, from the identities in (2.32), we can alternatively formulate smooth vector fields as derivations on smooth $C^\infty(M)$ functions, by permitting the p to vary in M.

This is formalized in the notion of "derivation on $C^\infty(M)$", namely an application

$$\boldsymbol{v} : C^\infty(M) \rightarrow C^\infty(M) \tag{2.44}$$

that:

i) It is $\mathbb{R}$-linear.

ii) It satisfies the Leibniz rule: $\boldsymbol{v}(fg) = \boldsymbol{v}(f)g + f\boldsymbol{v}(g)$.

Given (2.32), the ensemble of "derivations on $C^\infty(M)$", denoted by $\mathfrak{X}(M)$, coincides with $\mathcal{T}M$. Defining the commutator:

$$[\boldsymbol{v}, \boldsymbol{w}](f) = \boldsymbol{v}(\boldsymbol{w}(f)) - \boldsymbol{w}(\boldsymbol{v}(f)) \quad , \quad \forall f \in \mathbb{C}^\infty(M) \ , \tag{2.45}$$

$[\cdot, \cdot]$ satisfies:

[8] This $C^\infty(M)$-linearity is inherited from the definition (2.40) and the $\mathbb{R}$-linearity of vectors acting on 1-forms at p. By duality, the space $\mathcal{T}M$ is $C^\infty(M)$-linear, but note that this does not define a vector space, but rather a $C^\infty(M)$-module, since $C^\infty(M)$ is a ring, but not a field (as $\mathbb{R}$).

i) $\mathbb{R}$-bilinearity.

ii) Skew-symmetry.

iii) Jacobi identity.

This confers $\mathfrak{X}(M)$ with the structure of an (infinite-dimensional) Lie algebra.

As indicated above, the equivalence between a vector field $\boldsymbol{v}$ seen as a C^∞-linear application on 1-forms and as a $\mathbb{R}$-linear derivation on C^∞ functions follows from the upgadre of Eq. (2.32) to the field level, namely

$$\boldsymbol{v}(df) = df(\boldsymbol{v}) = \boldsymbol{v}(f) \ . \tag{2.46}$$

2.3.2 Vector fields as dynamical systems: integral curves and vector flows

A vector field $\boldsymbol{v}$ assigns smoothly a vector $\boldsymbol{v}_p$ to each point of the manifold. We can consider a curve $\gamma : I \to M$ such that its velocity, that is it tangent vector $v^a = d\gamma^a d\lambda$, coincides at each point $p = \gamma(\lambda)$ of the curve with the vector $\boldsymbol{v}_p$ of $\boldsymbol{v}$ at that point. Such a curve, tangent to $\boldsymbol{v}$ is called an integral curve of the vector field.

If we consider a coordinate chart, $U, \{x^a\}$, so that the vector field is written as $\boldsymbol{X} = X^a\partial_a$, the curve γ with local parametrization $x^a(\lambda)$ is an integral curve of $\boldsymbol{x}$ if it satisfies

$$\frac{dx^a}{d\lambda} = X^a \ . \tag{2.47}$$

This defines in the open set $\tilde{U} \subset \mathbb{R}^n$ a system of ordinary differential equations, that is, a dynamical systems. Vector fields are therefore dynamical systems on manifolds.

We assume that the solution (2.47) can be extended to the whole manifold M. This provides with the following mapping

$$\begin{array}{rcl} F : I \subset \mathbb{R} \times M & \to & M \\ (\lambda, x) & \mapsto & F_\lambda(x) \end{array} \tag{2.48}$$

such that $\gamma_{x_o}(\lambda) := F_\lambda(x_0)$ is an integral curve starting at x_0, that is $\gamma_{x_o}(0) := F_0(x_0) = x_0$. The one-parametic flow mapping F can be seen from two perspectives:

i) Fixing x_0 it provides the integral curve to $\boldsymbol{X}$ starting at x_0.

ii) Fixing λ it sends points $x \in M$ to points $F_\lambda(x)$. Since the vector field assigns a unique vector at each point, F_λ provides a (local) diffeomorphism (as long as it does not vanish, where critical point of the dynamical system occur).

In the sense ii), vector fields $\boldsymbol{v}$ can be seen as infinitesimal diffeomorphisms. Together with the derivation commutator (2.45), this constitutes the Lie algebra of local diffeomorphisms.

The one-parametric flow F_λ in (2.48) will play a key role later when introducing the notion of Lie derivative. Although it would natural, from a structural mathematical perspective to introduce such a notion here, we will introduce when discussing the symmetries of a spacetime.

2.4 The metric tensor. Metric type of vectors.

The spacetime is more than the collection of occurring physical "events". It encodes fundamentally the notions of standard "clocks" and "rules". The latter entail the notion of "distance", either in "time" or in "space" and are therefore metric notions. The spacetime must be therefore endowed, at each point, with a structure capable of determining which are the spacelike directions, the timelike directions and the directions followed by light rays, as well as spatial and time *distances* between events.

Following the model of special relativity, this is accomplished by introducing an additional structure to the differentiable manifold M, namely a (non-degenerate) Lorentzian metric tensor $\boldsymbol{g}$. A spacetime is then given by the couple $(M, \boldsymbol{g})$.

2.4.1 Metric "tensor"

In the next chapter we will discuss systematically the general notion of a tensor. At this stage, we just introduce a geometric object that, at each point $p \in M$, provides a quadratic form $\boldsymbol{g}_p$ on T_pM in such a way that this assignment is smooth. Following the model of vector fields and one-forms, we upgrade the $\mathbb{R}$-bilinear T_pM form to a C^∞-bilinear form on $\mathcal{TM}$. That is, we introduce a mapping

$$\begin{aligned} \boldsymbol{g} : \mathcal{TM} \times \mathcal{TM} &\rightarrow C^\infty(M) \\ (\boldsymbol{v}, \boldsymbol{w}) &\mapsto \boldsymbol{g}(\boldsymbol{v}, \boldsymbol{w}) \ . \end{aligned} \tag{2.49}$$

C^∞-bilinear in both entries. In the next chapter we will refer to $\boldsymbol{g}$ as 2-times covariant tensor. In order $\boldsymbol{g}$ to represent a metric field, we require

i) It is symmetric: $\boldsymbol{g}(\boldsymbol{v}, \boldsymbol{w}) = \boldsymbol{g}(\boldsymbol{w}, \boldsymbol{v})$, $\forall \boldsymbol{v}, \boldsymbol{w} \in \mathcal{TM}$.

ii) It is non-degenerate: if $\boldsymbol{v}$ is such that $\boldsymbol{g}(\boldsymbol{v}, \boldsymbol{v}) = 0$, $\forall \boldsymbol{w}$, then $\boldsymbol{v} = 0$.

Given two one-forms $\alpha, \beta \in \mathcal{T}^*\mathcal{M}$, characterized as $C^\infty(M)$ linear mappings $\alpha, \beta : \mathcal{TM} \rightarrow \mathbb{R}$, we can introduce the tensor product $\alpha \otimes \beta$ (to be generalized in next chapter to general tensors) as

$$\alpha \otimes \beta : \mathcal{TM} \times \mathcal{TM} \rightarrow C^\infty(M) \tag{2.50}$$

such that

$$\alpha \otimes \beta(\boldsymbol{v}, \boldsymbol{w}) = \alpha(\boldsymbol{v})\beta(\boldsymbol{w}) \quad , \quad \forall \boldsymbol{v}, \boldsymbol{w} \in \mathcal{TM} \ , \tag{2.51}$$

the $C^\infty(M)$-linearity of α and β guarantees the $C^\infty(M)$-bilinearity of $\alpha \otimes \beta$. Moreover, given a basis $\{\omega^a\}$ of $\mathcal{T}^*\mathcal{M}$, a basis of $C^\infty(M)$-bilinear mappings (2.49) is provided by the tensor products $\{\omega^a \otimes \omega^b\}$. In particular, choosing a chart $(U, \{x\}$ we have seen that the set of forms $\{\omega^a = dx^a\}$ provide a basis for one-forms. Therfore, in this local chart, the metric $\boldsymbol{g}$ can be written as

$$\boldsymbol{g} = g_{ab} dx^a \otimes dx^b \tag{2.52}$$

for C^∞ functions g_{ab}. In this chart, the metric conditions above translate in:

i) The matrix g_{ab} is symmetric: $g_{ab} = g_{ba}$.

ii) The matrix g_{ab} is non-degenerate. In particular, $det(g_{ab}) \neq 0$ and, at each point, the inverse matrix $(g_{ab})^{-1}$ exists. We shall denote that inverse as g^{ab}, so

$$g^{ab} g_{bc} = g_{cb} g^{ba} = \delta^a{}_b \ . \tag{2.53}$$

Raising and lowering of indices.

The metric $g_{\mu\nu}$ provides a canonical isomorphism between TM and T^*M, not depending on coordinates. Indeed, given its non-degenerate character we consider the *metric* tensor on $T*M$ whose component expression, denoted as $g^{\mu\nu}$ is given by the inverse matrix of $g_{\mu\nu}$. That is

$$g^{\mu\rho}g_{\rho\nu} = g_{\nu\rho}g^{\rho\mu} = \delta^{\mu}{}_{\nu} \tag{2.54}$$

Then, given a contravariant vector V^μ and a covariant vector α_μ, we construct the associated covariant and covariant vectors, respectively, as

$$V_\mu \equiv g_{\mu\nu}V^\nu \quad , \quad \alpha^\mu \equiv g^{\mu\nu}\alpha_\nu \tag{2.55}$$

This operations are usually referred to as *lowering* and *raising* indices.

2.4.2 Lorentzian signature: vector metric types and light cone.

Lorentzian signature

The symmetric tensor $\boldsymbol{g}$ can be diagonalized at each point $p \in M$. In particular, at each T_pM, a basis of vector can be chosen such that the metric is diagonal with only 1 or 1 in the diagonal (the non-degeneracy conditions guarantee that there are no zeros in the diagonal). If the metric can be taken at each point of M to the form

$$\boldsymbol{g}_p = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} , \tag{2.56}$$

we say that is of Lorentzian type. We say that it has Lorentzian signature that we write sign$(\boldsymbol{q}) = (-1\,1\ldots1)$.

Note that there is an equivalent choice as sign$(\boldsymbol{g}) = (1-1\ldots-1)$. Both of them are natural in different settings, namely the $(-11\ldots1)$ if we want to make contact with the Riemann geometry of "space-slices" and $(-11\ldots1)$ if one is focusing trajectories of particles (as it is the case usually in high-energy physics) or in the spinorial approach to General Relativity. Here we will stick to the $(-11\ldots1)$ convention.

As a non-denerate 2-times covariant tensor, the metric $\boldsymbol{g}$ provides an isomorphism between $\mathcal{T}M$ and $\mathcal{T}^*M$. This is referred to as raising and lowering indices in the relativity literature

$$\begin{array}{rcl} \sharp : \mathcal{T}^*M & \rightarrow & \mathcal{T}M \\ \boldsymbol{\alpha} & \mapsto & \boldsymbol{\alpha}^\sharp \end{array} \tag{2.57}$$

such that

$$\boldsymbol{g}(\boldsymbol{\alpha}^\sharp, \boldsymbol{v}) = \boldsymbol{\alpha}(\boldsymbol{v}) \ , \ \forall \boldsymbol{v} \in \mathcal{T}M \tag{2.58}$$

and

$$\begin{array}{rcl} \flat : \mathcal{T}M & \rightarrow & \mathcal{T}^*M \\ \boldsymbol{v} & \mapsto & \boldsymbol{v}^\flat \end{array} \tag{2.59}$$

such that

$$\boldsymbol{v}^\flat(\boldsymbol{\alpha}) = \boldsymbol{g}(\boldsymbol{v}, \boldsymbol{w}) \ , \ \forall \alpha \in \mathcal{T}^* M \tag{2.60}$$

In particular, in a coordinate basis it follows

$$\begin{aligned} \boldsymbol{v} &= v^a \partial_a \to \boldsymbol{v}^\flat = v_a dx^a \ , \ \text{with} v_a = g_{ab} v^b \\ \boldsymbol{\alpha} &= \alpha_a dx^a \to \alpha^\sharp = \alpha^a \partial_a \ \text{with} \alpha^a = g^{ab} \alpha_b \end{aligned} \tag{2.61}$$

Light cone

The squared-norm of a vector $\boldsymbol{v}$ at a point $p \in M$ is given by

$$v^2 = \boldsymbol{g}(\boldsymbol{v}, \boldsymbol{v}) = g_{ab} v^a v^b = v^a v_b \tag{2.62}$$

The Lorentzian nature of $\boldsymbol{g}$ permit to classify the vectors at $T_p M$ in three cathegories:

i) Spacelike vectors: $g_{ab} v^a v^b > 0$.

ii) Timelike vectors: $g_{ab} v^a v^b < 0$.

iii) Lightlike or null vectors: $g_{ab} v^a v^b = 0$.

Therefore, the Lorentzian structure of the spacetime permits to introduce at each point p the notion of *light cone*, as the set of vectors in $T_p M$ of zero norm.

Light or null curves move along light cones in trajectories with null derivative vector. Particles moving at a speed smaller that light velocity lay inside the light cones, with timelike derivatives. Finally, particle moving faster than light, or simply curves joining points that are *simultaneous* in some coordinate system, have spacelike derivatives. Null and timelike curves are referred as causal.

The light cone separates $T_p M$ in three parts: two non-connected interior regions with timelike vectors, a connected (if $dim(M) \geq 3$) exterior region formed by spacelike vectors, and the light cone itself with null vectors. In particular, at each point $p \in M$ one assigns a *future* and *past* character, respectively, to each one of the connected components of the timelike region, as well as its component of the light cone.

[**Figure lightcone**]

2.4.3 Some basic causal notions

. The structure given by the emsemble of light cones determines the causal structure of spacetime. In particular, in this causality context it is natural to require that a global assignment of *future* and *past* can be consistently introduced. If such an assignment is possible, the spacetime $(M, \boldsymbol{g})$ is said to be *time orientable*.

[**Non-orientabl spacetime**]

Lemma 1 (Time orientability). *If $(M, \boldsymbol{g})$ is time orientable, then there exists a (global) smooth nonvanishing timelike vector field $\boldsymbol{t} \in \mathfrak{X}(M)$.*

2.4.4 Measuring proper distances and proper times: element of line.

The light cone structure of the spacetime allows us to structurate the spacetime in spacelike, timelike and lightlike directions. But the metric has more structure (actually very little more, just a scale), permitting us to measure distances spacelike curves and time intervals along timelike curves. This is provided by the notion of *element of line* associated to the metric in a given coordinate system, simply a quadratic form on infinitesimal displacements in spacetime:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\mu \tag{2.63}$$

This can be seen as a generalization of Pythagoras theorem for infinitesinal triangles.

If we consider a spacelike curve $\gamma(\lambda)$ parametrized by λ in coordinates $\{x^\mu\}$, i.e. $(x^\mu(\lambda))$, the evaluation of (2.63) on $\gamma(\lambda)$ gives

$$ds^2 = g_{\mu\nu}(\gamma(\lambda)) \frac{dx^\mu}{d\lambda} \frac{dx^\mu}{d\lambda} d\lambda^2 \quad . \tag{2.64}$$

For a spacelike curves the *arc length* can be simply written as

$$ds = \sqrt{g_{\mu\nu}(\gamma(\lambda)) \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda \tag{2.65}$$

With our convention for the spacetime signature $(-1, 1, 1, 1)$, the element of proper time along timelike curves is given by $-c^2 d\tau = ds^2$, that is

$$d\tau = \frac{1}{c} \sqrt{\left| g_{\mu\nu}(\gamma(\lambda)) \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \right|} d\lambda \tag{2.66}$$

2.4.5 Observers.

An observer in General Relativity is provided by a timelike curve γ whose 4-velocity u^μ is normalized to -1, that is

$$u^\mu = \frac{d^\mu}{d\lambda} \quad , \quad u^\mu u_\mu = g_{\mu\nu}(\gamma(\lambda)) \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \quad = -1 \tag{2.67}$$

Using (2.66) we can write $u^\mu = \frac{dx^\mu}{d\tau}$.

2.5 Minkowski spacetime. Rindler coordinates

The first spacetime we have encountered corresponds to the one in special relativity, corresponding to the absence of gravity. Its line element in coordinates corresponding to an inertial frame

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2 \tag{2.68}$$

Note that Poincaré trasnformations (2.6) preserve the form of this line element. They are the first example of isometries. The Minkowski geometry illustrates some of the points in this lecture. First, note that parametrizing a timelike curve by $\lambda = ct$, proper time writes

$$d\tau = \sqrt{1 - \frac{1}{c^2} \frac{d\vec{x}}{dt} \cdot \frac{d\vec{x}}{dt}} dt \tag{2.69}$$

and an observer

$$u^\mu = \left(\gamma, \gamma \frac{d\vec{x}}{dt}\right) \tag{2.70}$$

with $\gamma = dt/d\tau = 1/\sqrt{1 - \frac{1}{c^2}\frac{d\vec{x}}{dt} \cdot \frac{d\vec{x}}{dt}}$.

2.5.1 Absence of geometric meaning in the coordinates

We have insisted in the absence of a priori geometric meaning in the coordinates in General Relativity. This is illustrated byy the following example. Consider the metric with line element

$$ds^2 = -\frac{1}{t^2}dt^2 + dx^2 + dy^2 + dz^2 \tag{2.71}$$

for $0 < t < \infty$, $-\infty < x < \infty$, $-\infty < y < \infty$, $-\infty < z < \infty$. This suggests a metric with a bad behaviour as one approaches $t = 0$, possibly indicating some geometric non-trivial behaviour in its vicinity. However, if we make the transformation of variables

$$t' = \ln t \ , \ x' = x \ , \ y' = y \ , \ z' = z \tag{2.72}$$

with $-\infty < t' < \infty$, $-\infty < x' < \infty$, $-\infty < y' < \infty$, $-\infty < z' < \infty$. we realize that the metric can be written as

$$ds^2 = -c^2 dt'^2 + dx'^2 + dy'^2 + dz'^2 \tag{2.73}$$

where we recognize the familiar Minkowski spactime. We conclude that coordinates (t, x, y, z) are just *labels* without any intrinsic meaning.

Remarks.

- i) The expression *choice of coordinates* actually refers to the freedom in choosing the functional form of four of the functions in the set of ten functions $g_{\mu\nu}(x)$, where $\{x^\mu\}$ are just formal labels without meanings.

  In general, for a given metric the remaining six functions cannot be freely chosen (the freedom in coordinate choice is exhausted). In particular the question of under which conditions a line element can be trasnformed to the form (2.68), implies the resolution of an overdetermined system of partial differential equations, so that in general has no solution. Only in special cases satisfying certain *integrability conditions* the system can be solved. As we will comment later, this integrability conditions are given in terms of a tensorial quantity, precisely the *curvature tensor*. In other words, the gravitational field.

- ii) The singular behaviour in the metric functions can be due to two reasons: a) an actual singularity in the metric, b) a pathologic behaviour of the coordinates. Deciding with is the case is not always easy. The Rindler metric provides below a paradigmatic example of this that illustrates the behaviour that we will find in black holes.

2.5.2 Rindler spacetime

Let us now consider a more interesting example referred to as Rindler spacetime.

Let us consider the line element in a chart $\tilde{U}_1 = \{t, x\}$ that covers the whole spacetime $(M, \boldsymbol{g})$

$$ds^2 = -x^2 dt^2 + dx^2 \quad , \quad x \in]0, \infty[\ , \ t \in]-\infty, \infty[\ . \tag{2.74}$$

As we see, something seems to wrong with metric at $x = 0$, where the determinant vanishes and the metric is not invertible. This $x = 0$ seems as a "border" of spacetime through which we cannot get. Is this a geometric intrinsic issue or is it an artifact of the coordinate description?

Let us recast this metric in a different form. We are going to look for coordinates adapted to the geometry of the problem, namely to null directions. For this we consider curves $\gamma : I \to M$ that in the local chart is is written as $\lambda \mapsto (t(\lambda), x(\lambda))$. The correspondinf tangent vector

$$u^a = \left(\frac{dt}{d\lambda}, \frac{dx}{d\lambda}\right) \ . \tag{2.75}$$

Enforcing γ to be null at every point amounts to impose, usinf the line element (2.74)

$$u^a u_a = 0 \Leftrightarrow -x^2 \left(\frac{dt}{d\lambda}\right)^2 + \left(\frac{dt}{d\lambda}\right)^2 = 0 \tag{2.76}$$

That is

$$\left(\frac{dt}{d\lambda}\right)^2 = x^2 \left(\frac{dt}{d\lambda}\right)^2 \ , \tag{2.77}$$

from which we can write (assuming, e.g. $\frac{dt}{d\lambda} > 0$)

$$\left(\frac{dt}{dx}\right)^2 = \frac{1}{x^2} \ , \tag{2.78}$$

that leads to two solutions

$$\frac{dt}{dx} = \pm\frac{1}{x} \Rightarrow \left\{ \begin{array}{ccc} t & = & \ln x + u \\ t & = & -\ln x + v \end{array} \right. \tag{2.79}$$

where u and v are integration constants. At this point note that a given point in the chart $U \subset M$ can be labelled uniquely both by the coordinates $\{t, x\}$, but also by the pair of labels $\{u, v\}$, since the intersection of $u = \text{constant}_1$ and $v = \text{constant}_2$ univoquely defined the point. That is we can introduce another chart $\tilde{U}_2, \{u, v\}$ with coordinate change ("transition functions" φ_{12} and φ_{21} given by

$$\left\{ \begin{array}{ccc} u & = & t - \ln x \\ v & = & t + \ln x \end{array} \right. \ , \ \left\{ \begin{array}{ccc} t & = & \frac{1}{2}(u+v) \\ x & = & e^{\frac{v-u}{2}} \end{array} \right. \ , \ \begin{array}{c} u \in]-\infty, \infty[\\ v \in]-\infty, \infty[\end{array} \tag{2.80}$$

From this we can write the respectice relation between the coordinate basis for 1-forms

$$\left\{ \begin{array}{ccc} du & = & dt - \frac{1}{x}dx \\ dv & = & dt + \frac{1}{x}dx \end{array} \right. \ , \ \left\{ \begin{array}{ccc} t & = & \frac{1}{2}(du+dv) \\ x & = & \frac{1}{2}e^{\frac{v-u}{2}}(du-dv) \end{array} \right. \tag{2.81}$$

Inserting this into (2.74) we get

$$ds^2 = -e^{v-u} du dv \ , \tag{2.82}$$

At this point we can consider a second change of variables (that can be actually justified in geometric terms as looking for an “affibe parameter”, cf. [10]), to a new chart $\tilde{U}_3, \{U, V\}$, with coordinate change

$$\begin{cases} U &= -e^{-u} \\ V &= e^{v} \end{cases} \ , \quad \begin{matrix} U \in]-\infty, 0[\\ V \in]0, \infty[\end{matrix} \ . \tag{2.83}$$

(Notice that we could have chosen to place the “minus” sign $-$ differently, see below). Under this change for coordinates we get

$$dU = e^{-u} du \ , \ dV = e^{v} dv \ , \tag{2.84}$$

so that we finally get

$$ds^2 = -e^{v-u} du dv = -dU dV \quad , \quad -\infty < U < 0 \ , \ 0 < V < \infty. \tag{2.85}$$

If we look at the metric in this form, we see that there is no problem in the limits $U \to 0$ and $V \to 0$. Actually we can “extend” the metric to an enlarged range of the variables $\{U, V\}$. Namely, we can “enlarge” the spacetime M to $\tilde{M}$, the latter being now covered by a new chart $\tilde{U}_4, \{U', V'\}$ with

$$\begin{cases} U' &= U \\ V' &= V \end{cases} \ , \quad -\infty < U' < \infty \quad , \quad -\infty < V' < \infty \ . \tag{2.86}$$

At this point, we can introduce a final chart $\tilde{U}_5, \{T, X\}$ determined by the relations

$$\begin{cases} T &= \frac{1}{2}(U+V) \\ X &= \frac{1}{2}(U-V) \end{cases} \ , \quad \begin{cases} U &= T - X \\ V &= T + X \end{cases} \ , \quad \begin{matrix} T \in]-\infty, \infty[\\ X \in]-\infty, \infty[\end{matrix} \tag{2.87}$$

When writing the (extended) line element in this coordinates we find

$$ds^2 = -dT^2 + dX^2 \quad , \quad T \in]-\infty, \infty[\ , \ t \in]-\infty, \infty[\ . \tag{2.88}$$

that we recognize as the Minkowski spacetime $(\mathbb{M}^{\not\vDash}, \boldsymbol{eta})$ in dimension 2.

Rindler spacetime as a subset of Minkowski

The use of primes in (2.86) serves to emphasize that the ranges of the variables have in extended. Admittedly, such notation is a bit pedantic and we shall omit the primes in the following, just paying attention to keep track of the appropriate ranges.

Let us map the Rindler spacetime into the Minkowski spacetime. For this, let us explicitly find the coordinate change between charts $\tilde{U}_1, \{t, x\}$ and $\tilde{U}_5, \{T, X\}$. We write first the coordinate changes we have introduce above

$$\varphi_{21} : \begin{cases} t &= \frac{1}{2}(u+v) \\ x &= e^{\frac{v-u}{2}} \end{cases} \ , \ \varphi_{23} : \begin{cases} U &= -e^{-u} \\ V &= e^{v} \end{cases} \ , \ \varphi_{53} : \begin{cases} U &= T - X \\ V &= T + X \end{cases} \tag{2.89}$$

From this, we first try to express x as a function of (T, X). For this we write

$$x = (-VU)^{\frac{1}{2}} = (X^2 - T^2)^{\frac{1}{2}} \tag{2.90}$$

Analogously, using $u = -\ln(-U)$ and $v = \ln V$, we can write

$$\begin{aligned} t &= \frac{1}{2}(-\ln(-U) + \ln V) = \frac{1}{2}\ln\left(-\frac{V}{U}\right) = \frac{1}{2}\ln\left(\frac{X+T}{X-T}\right) \\ &= \frac{1}{2}\ln\left(\frac{1+T/X}{1-T/X}\right) = \tanh^{-1}\left(\frac{T}{X}\right) \end{aligned} \tag{2.91}$$

From this, we can conclude

$$\varphi_{51} : \begin{cases} t &= \tanh^{-1}\left(\frac{T}{X}\right) \\ x &= (X^2 - T^2)^{\frac{1}{2}} \end{cases} \tag{2.92}$$

To construct the inverse change, we just notice

$$x^2 = X^2 - T^2 \iff 1 = \left(\frac{X}{x}\right)^2 - \left(\frac{T}{x}\right)^2 , \tag{2.93}$$

from which we can write, for some $\alpha \in \mathbb{R}$

$$\frac{X}{x} = \cosh\alpha \quad , \quad \frac{T}{x} = \sinh\alpha \; , \tag{2.94}$$

from which

$$\tanh\alpha = \frac{T}{X} \Leftrightarrow \alpha = \tanh^{-1}\left(\frac{T}{X}\right) \; , \tag{2.95}$$

and from (2.91) we conclude $\alpha = t$. Putting together the coordinate changes, we can write

$$\varphi_{51} : \begin{cases} t &= \tanh^{-1}\left(\frac{T}{X}\right) \\ x &= (X^2 - T^2)^{\frac{1}{2}} \end{cases} \quad , \quad \varphi_{15} : \begin{cases} X &= x\cosh t \\ T &= x\sinh t \end{cases} \quad , \tag{2.96}$$

so we see that the change $(t, x) \to (T, X)$ is just a change into "hyperboloidal polar" coordinates in which the role of the "radius" is played by the position x and the "angle" is the time label "t". Intuition in the geometry of the problem is gained by comparing how the coordinate lines $x =$ constant and $t =$ constant.

Make Figure

Interpreting Rindler spacetime

More geometric intuition, and as a first practice in manipulated vector fields, is git from the coordinate vector fields in each chart.

We focus here on ∂_t. Note first that no coordinate function of $\boldsymbol{g}$ depends on x so, in the sense that $\partial_t g_{ab} = 0$, this vector field leaves the metric invariant and we refer to it as an infitemisal isometry. We will formalise this notion later in terms of the notion of Lie derivative.

Then, let us compare how the coordinate vector fields in the different coordinates relate to each other. pplying the change rule to the coordiante transformations above, we can write

$$\partial_t = \partial_u + \partial_v = -U\partial_U + V\partial_V \tag{2.97}$$

From here, we finally obtain

$$\partial_t = T\partial_X + X\partial_T \ . \tag{2.98}$$

This expression is particularly illuminating since, as we will see when introducing infinitesimal isometries, it corresponds to a infitesimal generator of a "boost" along the X direction. It is illustrative to solve the associated dynamical system defined as

$$\frac{dx^a}{d\lambda} = (\partial_t)^a \ , \tag{2.99}$$

that, in (T, X) coordinates write

$$\begin{aligned} \frac{dT}{d\lambda} &= X \\ \frac{dX}{d\lambda} &= T \ , \end{aligned} \tag{2.100}$$

that can be written as a second order differential equation

$$\frac{d^2T}{d\lambda^2} = T \iff \frac{d^2T}{d\lambda^2} - T = 0 \ , \tag{2.101}$$

with solutions

$$T = X_0 \cosh\lambda + T_0 \sinh\lambda \ , \tag{2.102}$$

for constants X_0 and T_0, this implying

$$X = X_0 \sinh\lambda + T_0 \cosh\lambda \ , \tag{2.103}$$

so we can write

$$\begin{pmatrix} X \\ T \end{pmatrix} = \begin{pmatrix} \cosh\lambda & \sinh\lambda \\ \sinh\lambda & \cosh\lambda \end{pmatrix} \begin{pmatrix} X_0 \\ T_0 \end{pmatrix} \tag{2.104}$$

corresponding precisely to a boost transformation where λ is actually a "celerity" α related to the boost . This indicates that the trajectories associated with the integral curves of the vector ∂_t, that in (t, x) just stay at a fix x_0, correspond from the Minkowski perspective to trajectories of increasing velocity, since λ grows without bound.

Let us push a bit further the intuition on this accelerated motion and therefore, on the interpretation of this "spacetime" from a physical perspective. For this we consider the notion of observer introduced above, adapted to the integral curves of ∂_t. First of all we notice that $t^a\partial_t$ is indeed timelike (indeed, $\boldsymbol{g}(\partial_t, \partial_t) = -x < 0$) so that we can define an observer associated with the fixed $x = x_0$ position

$$u^a = \frac{1}{\sqrt{|t^a t_a|}} t^a = \frac{1}{\sqrt{|g_{tt}|}} t^a = \frac{1}{\sqrt{x_0^2}} t^a = \frac{1}{x_0} t^a \ , \tag{2.105}$$

Since we can write

$$u^a = \frac{dx^a}{d\tau}\ , \tag{2.106}$$

we can write

$$\frac{dt}{d\tau} = \frac{1}{x}\ ,\quad \frac{dx}{d\tau} = 0\ , \tag{2.107}$$

so that the solution to the curve along the ∂_t vector is

$$(t = \frac{1}{x_0}\tau, x = x_0) \tag{2.108}$$

In particular this means that the α parameter associated with boost velocity v by $v = \tanh\alpha$, is given by $\alpha = t = \frac{1}{x_0}\tau$. This gives a loose notion that $\frac{1}{x_0}$ actually provides a notion of acceleration, and the latter is constant. To be more precise, if we evaluate the relativistic velocity u^a in (T, X) coordinates along a curve (2.108), using (2.96) to get

$$\begin{aligned} u^T &= \frac{dT}{d\tau} = \cosh\left(\frac{\tau}{x_0}\right) \\ u^X &= \frac{dX}{d\tau} = \sinh\left(\frac{\tau}{x_0}\right)\ , \end{aligned} \tag{2.109}$$

and then we calculate the relativistic acceleration by taking a second derivative a^a [9] along τ, as $a^a = \frac{du^a}{d\tau}$, we get

$$\begin{aligned} a^T &= \frac{du^T}{d\tau} = \frac{1}{x_0}\sinh\left(\frac{\tau}{x_0}\right) \\ a^X &= \frac{du^X}{d\tau} = \frac{1}{x_0}\cosh\left(\frac{\tau}{x_0}\right)\ . \end{aligned} \tag{2.110}$$

If we compute now the spacetime "norm" we get

$$\begin{aligned} \boldsymbol{g}(\boldsymbol{a}, \boldsymbol{a}) &= g_{ab}a^a a^b = a^a a_a = -\left(\frac{1}{x_0}\sinh\left(\frac{\tau}{x_0}\right)\right)^2 + \left(\frac{1}{x_0}\cosh\left(\frac{\tau}{x_0}\right)\right)^2 \\ &= \left(\frac{1}{x_0}\right)^2\left(\cosh^2\left(\frac{\tau}{x_0} - \sinh^2\left(\frac{\tau}{x_0}\right)\right) = \left(\frac{1}{x_0}\right)^2 > 0 \end{aligned} \tag{2.111}$$

So we find that a^a is a spacelice vector (as it should, since it must be orthogonal to the normalised relativistic velocity), and its norm is constant (on a given curve!)

$$|\boldsymbol{a}| = \frac{1}{x_0} \tag{2.112}$$

An interpretation of such trajectories is that they correspond to uniformly accelerated observers in Minkowski, and the "Rindler spacetime" to the description of Minkowski from the perspective of such obsercers.

Remark 1: Remarks about Rindler spacetime: to further develop

[9] An important warning: this is an operation we have not yet defined in general, namely the derivative of a vector. In this particular flat case, and in this coordinates (T, X), this is however a legitimate calculation as we will see once we will introduce connections.

i) Spacetime as seen by an accelerated observer.

ii) Rindler is a wall of light moving to the right, and an observer excaping from it at a constant acceleration.

ii) One can cover the whole Minkowski with other “Rindler charts”, by adapting appropriately the signs in the coordinate changes. **[Exercise!]**

- Resemblance to Schwarzschild, from a metric perspective.
- Interpretation from the “equivalence principle”: on the “constant” gravitational field and Bell’s acceleration.
- Open project: relate to Unruh effect.
- Notion of horizon: causal disconnection.

2.6 Basic causality notions

Chapter 3

Gravity as spacetime curvature II: tensors, connections, curvature

Contents

3.1 Tensors

We introduce tensors, namely the tools to write equations that ta well defined point-wise and expressed in a manner explicitly independent of the chose coordinates: if the equation is valid a coordinate system is valid in any other. Introduction of derivatives of tensors. Connections and curvature.

3.1.1 Tensor fields

Taking a step further, we can define the space $\mathcal{T}^n_m(M)$ of n-times contravariant and m-covariant tensor fields as the ensemble of $C^\infty(M)$-multilinear smooth applications

$$\boldsymbol{T} : \mathcal{T}^*M \times \ldots^{n)} \times \mathcal{T}^*M \times \mathcal{T}M \times \ldots^{m)} \times \mathcal{T}M \to C^\infty(M) \ . \tag{3.1}$$

$\mathcal{T}^n_m(M)$ is also denoted as $\binom{n}{m}$. We note that $\mathcal{T}M = \mathcal{T}^1_0(M)$ and $\mathcal{T}^*M = \mathcal{T}^0_1(M)$. Using the notion of tensor product (over the module $C^\infty(M)$), we can write $\mathcal{T}^n_m(M)$ as

$$\mathcal{T}^n_m(M) = \mathcal{T}M \otimes \ldots^{n)} \mathcal{T}M \otimes \mathcal{T}^*M \times \ldots^{m)} \mathcal{T}^*M = \mathcal{T}M^{\otimes n} \otimes \mathcal{T}^*M^{\otimes m} \tag{3.2}$$

This characterization has the advantage of providing directly a local chart basis in $\mathcal{T}^n_m(M)$, in terms of tensor products of the bases in (2.43). In brief, we can write

$$\boldsymbol{T} = T^{\mu_1\mu_2\ldots\mu_n}{}_{\nu_1\nu_2\ldots\nu_m} \partial_{\mu_1} \otimes \partial_{\mu_2} \ldots \otimes \partial_{\mu_n} \otimes dx^{\nu_1} \otimes dx^{\nu_2} \ldots \otimes dx^{\nu_m} \quad , \quad \forall \boldsymbol{T} \in \mathcal{T}^n_m(M) \ . \tag{3.3}$$

This permits us to write the transformation rule of tensors under a change of coordinates. If we write in two coordinate systems

$$\mathbf{T} = T^{i_1\ldots i_n}{}_{j_1\ldots j_m} \frac{\partial}{\partial x^{i_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{i_n}} \otimes dx^{j_1} \otimes \ldots \otimes dx^{j_m}$$

and

$$\mathbf{T} = T'^{i_1 \dots i_n}{}_{j_1 \dots j_m} \frac{\partial}{\partial x'^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x'^{i_n}} \otimes dx'^{j_1} \otimes \dots \otimes dx'^{j_m} \ ,$$

then it follows from multilinearity

$$T'^{i_1 \dots i_n}{}_{j_1 \dots j_m} = \left(\frac{\partial x'^{i_1}}{\partial x^{k_1}}\right) \dots \left(\frac{\partial x'^{i_n}}{\partial x^{k_n}}\right) \left(\frac{\partial x^{l_1}}{\partial x'^{j_1}}\right) \dots \left(\frac{\partial x^{l_m}}{\partial x'^{j_m}}\right) T^{k_1 \dots k_n}{}_{l_1 \dots l_m} \ .$$

Tangent, cotangent and tensor bundle.

An alternative approach to tensor fields is captured in terms of the notion of *tangent bundle* over M. In particular, we consider a space formed by each point $p \in M$ (M is the *base* of the bundle) together with its tangent plane T_pM considered as a *fiber*. The ensemble formed by all theses pairs (p, T_pM) form an space (actually a manifold) referred to as the *tangent bundle* TM. Analogously one introduces the cotangent bundle T^*M as the union of all points together with their cotangent space. That is

$$TM = \bigcup_{p\in M} T_pM \quad , \quad T^*M = \bigcup_{p\in M} T_p^*M \ . \tag{3.4}$$

A bundle P has a natural projection to its base $\pi : P \to M$, so that every point in the bundle can be "vertically" projected to a point in M. In this setting, a vector field $\boldsymbol{v}$ is a smooth application from M to TM that sends (smoothly) each $p \in M$ to a point in its fiber (namely $\pi \circ \boldsymbol{v} = \mathrm{id}_M$). That is

$$\boldsymbol{v} : M \ \to \ TM \tag{3.5}$$

$$p \ \mapsto \ \boldsymbol{v}_p \in T_pM \ . \tag{3.6}$$

Such an application is called a smooth *section* of the tangent bundle TM. In particular, the set of vector fields $\mathcal{T}M$ introduced above is the set of sections of the bundle TM. Analogously, a *1-form* α is a smooth section in the cotangent bundle T^*M

$$\alpha : M \ \to \ T^*M \tag{3.7}$$

$$p \ \mapsto \ \alpha_p \in T_pM \ . \tag{3.8}$$

We can the consider the tensor bundle T^n_mM as

$$T^n_mM = TM \otimes \dots^{n)} \otimes TM \otimes T^*M \otimes \dots\dots^{m)} \otimes T^*M = TM^{\otimes n} \otimes T^*M^{\otimes m} \tag{3.9}$$

A n-times contravariant and m-times covariant tensor field is then a smooth section in the tensor bundle T^n_mM. The bundle formalism is the natural language to address global topological issues related to tensor fields. In our present setting we will not resort to its full strength [1].

[1]Still another approach to vector fields, and the $1-forms$ and tensor fields by duality and multilinearity, would be in terms of the notion of derivations of functions $C^\infty(M)$. Indeed, when looking at a vector $\boldsymbol{v}_p \in T_pM$ as a derivation on $C^\infty(M)$, $\boldsymbol{v}_p$ defines the $\mathbb{R}$-linear mapping

$$\boldsymbol{v}_p : \mathcal{F}(M) \to \mathbb{R} \ . \tag{3.10}$$

A vector field $\boldsymbol{v}$ provides a smooth rule to vary p on M, in such a way that when acting at each p on functions in $C^\infty(M)$ we obtain functions that are also smooth. In other words, a vector field is smooth $C^\infty(M)$-linear

Tensors as point-like and multi-linear objects.

Note that from the very construction tensors are point-like objects. In addition, their multi-linear character guarantees the following key property: If a tensor vanishes in a certain coordinate system, it vanishes in all coordinate systems.

Gradients, vectors, directional derivatives.

A generalization of the standard gradient ∇f of a function is provided by df. Contracting the gradient with a given vector V^μ, we construct the directional derivative along V^μ. The latter is given above by $V(f) = V(df)$. It is useful to introduce a notation in terms of the "nabla" operator

$$\begin{aligned} \nabla f &= df = \partial_\mu f dx^\mu = \nabla_\mu f dx^\mu & (3.13)\\ \nabla_V f &= V(df) = V(f) = V^\mu \partial_\mu(f) = V^\mu \nabla_\mu f & (3.14) \end{aligned}$$

where $\nabla_\mu f = \partial_\mu f$.

3.2 Exercises: tensor manipulation (indices gymnastics).

- Transformation rules of contravariant and covariant vectors under a coordinate transformation.
- Transformation of the metric tensor.
- Transformation of the volume element.
- Coordinate velocity: Is the coordinate velocity of light constant?
- Conformal structure and light cone structure: conformal transformations of the spacetime metric.

application

$$\boldsymbol{v} : C^\infty(M) \to C^\infty(M) \tag{3.11}$$

satisfying in addition the Leibniz-rule

$$\boldsymbol{v}(fg) = \boldsymbol{v}(f)g + f\boldsymbol{v}(g) \quad \forall f, g \in C^\infty(M) \ . \tag{3.12}$$

This defines $\boldsymbol{v}$ as a derivation in $C^\infty(M)$.

Chapter 4

Gravity as spacetime curvature III: Einstein equation

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4.1 Einstein equation

Einstein equation

Chapter 5

Schwarzschild Solution

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5.1 Derivation of Schwarzschild solution. Birkhoff's theorem

5.1.1 Resolution of Einstein equations: vacuum spherically symmetric case

[To be completed following MTW 32.2.] Result: there exits coordinates, adapted to spherical symmetry, in which the line element of vacuum spherically symmetric spacetime writes

$$ds^2 = -f(r)dt^2 + (f(r))^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right) . \tag{5.1}$$

with[1]

$$f = f(r) = \left(1 + \frac{C}{r}\right) . \tag{5.2}$$

5.1.2 Fixing the integration constant: Newton's theory of Gravity again

Let us fix the parameter C in (5.1). This point requires to make contact with Newton's theory of gravitation. This can be addressed (at least) in two approaches:

i) Field equations approach: to impose that the solution to linearized Einstein equations recover the solution to Poisson's equation for Newton's gravity.

ii) Particle-motion equation approach: to impose that, at large distaces $r \to \infty$, test-particles follow Keplerian motion dictated by Newton's Gravitation law.

We follow here the second approach (cf. e.g. [10] for the discussion of linearized gravity). Let us consider an observer with 4-velocity u^a in (5.1). First we write

$$u^\mu = \frac{dx^\mu}{d\tau} = (\dot{t}, \dot{r}, \dot{\theta}, \dot{\varphi}) . \tag{5.3}$$

[1]Note the physical dimensions of C: $[C]$ = Length.

We fix trajectories to the equatorial plane: $\theta = \frac{\pi}{2}, \dot{\theta} = 0$. We impose the arc-length normalization condition $u^a u_a = -1$

$$-1 = g_{ab} u^a u^b = -f\dot{t}^2 + \frac{1}{f}\dot{r}^2 + r^2 \underbrace{\sin^2\theta}_{=1} \dot{\varphi}^2 \tag{5.4}$$

At this point, in order to simplify the discussion and illustrate the usefulne

Lemma 1 (Conserved quantities along geodesics). *Given a Killing vector k^a and a geodesic with tangent vector u^a, the quantity*

$$k = k^a u_a \ , \tag{5.5}$$

is preserved along geodesics.

Proof. Indeed, we evaluate $\dfrac{dk}{d\tau}$ along geodesics

$$\frac{dk}{d\tau} = u^a \nabla_a k = u^a \nabla_a (k^b u_b) = u^a (\nabla_a u_b) u^b + k_b u^a \nabla_a u^b = 0 \ , \tag{5.6}$$

where the first term in the last equality vanishes from the Killing condition, whereas the second vanishes from the geodesic equation. □

Using lemma 1 for the Killing vectors $t^a = \partial_t$ (stationarity) and $\varphi^a = \partial_\varphi$ (rotation around the z-axis), we can defined the conserved quantities E (energy per mass) and L (angular momentum per mass)

$$\begin{aligned} E = -k^a u_a = -f\dot{t} \quad &\Longleftrightarrow \quad \dot{t} = -\frac{E}{f} \\ L = \varphi^a u_a = r^2 \dot{\varphi} \quad &\Longleftrightarrow \quad \dot{\varphi} = \frac{L}{r^2} \end{aligned} \tag{5.7}$$

Substituing into Eq. (5.4), we get the expression for the (square of the) energy of a timelike orbit

$$E^2 = \dot{r}^2 + f\left(1 + \frac{L^2}{r^2}\right) \ . \tag{5.8}$$

We focus for simplicity on radial geodesics, $L = 0$, so that

$$E^2 = \dot{r}^2 + f(r) \ . \tag{5.9}$$

Deriving this expression along geodesics, and using the constancy of E,

$$0 = 2\dot{r}\ddot{r} + \dot{r}\frac{df}{dr} \overset{\dot{r}=0}{\Longleftrightarrow} 2\ddot{r} + \frac{df}{dr} = 0 \quad ; \quad \ddot{r} = -\frac{f'}{2} \tag{5.10}$$

On the other hand, we can write (assuming that $dt/d\tau > 0$) and using (5.7)

$$\dot{r} = \frac{dr}{d\tau} = \frac{dr}{dt}\frac{dt}{d\tau} = \dot{t}\frac{dr}{dt} = -\frac{E}{f}\frac{dr}{dt} \ . \tag{5.11}$$

From this we can derive

$$\ddot{r} = \frac{E^2}{f}\left(-\frac{f'}{f^2}\left(\frac{dr}{dt}\right)^2 + \frac{1}{f}\frac{d^2 r}{dt^2}\right) \ . \tag{5.12}$$

In the non-relativistic limit $c \to \infty$, and at large distances, one can conclude (cf. Exercise 1)

$$\ddot{r} \sim \frac{d^2 r}{dt^2} \ . \tag{5.13}$$

Using then (5.10), together with $f'(r) = -\frac{C}{r^2}$ and Newton's Universal law of gravitation $\frac{d^2 r}{dt^2} = -\frac{M}{r}$, one concludes

$$C = -2M \ . \tag{5.14}$$

We can therefore write Schwarzschild line element as

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right) \ . \tag{5.15}$$

To reintroduce physical dimensions, cf. Exercise 1, one shifts $M \to \frac{GM}{c^2}$.

Exercise 1: Perfoming the shift $\tau \to c\tau$ and $t \to ct$, so that (5.1) writes $c^2 d\tau^2 = c^2 f(r) dt^2 - (f(r))^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right)$, reproduce the steps in the text to get:

$$\begin{aligned}
\dot{t} &= -\frac{E}{fc^2} \\
\frac{E^2}{c^2} &= f(r)c^2 + \dot{r}^2 = f(r)c^2 + \frac{E^2}{f^2 c^4}\left(\frac{dr}{dt}\right)^2 \\
0 &= f'\dot{r}c^2 + 2\dot{r}\ddot{r} \Longrightarrow \ddot{r} = -\frac{f'c^2}{2} \\
\ddot{r} &= \frac{E^2}{c^4 f}\left(-\frac{f'}{f^2}\left(\frac{dr}{dt}\right)^2 + \frac{1}{f}\frac{d^2 r}{dt^2}\right) \ .
\end{aligned}$$

In the non-relativistic limit $c \to \infty$, $\frac{dr}{dt} \ll c$, conclude from the second equation

$$\frac{E^2}{c^4 f} \sim 1 \ , \tag{5.16}$$

whereas from the third and the fourth

$$\frac{1}{f}\left(\frac{E^2}{c^4 f}\right)\frac{d^2 r}{dt^2} = -\frac{f'c^2}{2} - \frac{E^2}{c^4 f}\frac{f'}{f^2}\left(\frac{dr}{dt}\right)^2 \tag{5.17}$$

so that when $c \to \infty$

$$\frac{1}{f}\frac{d^2 r}{dt^2} = -\frac{f'c^2}{2} \tag{5.18}$$

Finally, using $f \sim 1$ at large distances and Newton's Universal law of gravitation $\frac{d^2 r}{dt^2} = -\frac{GM}{r^2}$, conclude from the form of $f(r)$, cf. (5.2)

$$C = -\frac{2GM}{c^2} \tag{5.19}$$

The ratio $\frac{G}{c^2}$, of dimensions $\left[\frac{G}{c^2}\right] = \frac{\text{Length}}{\text{Mass}}$, relates distances and masses in the theory through the compacity parameter Ξ introduced in (1.52). Its small value associates naturally a compact gravitational scale $\frac{G}{c^2}M$to a given mass M.

5.1.3 Birkhoff's theorem

From the derivation of Schwarzschild's line element, we can state the following theorem by Birkhoff (cf. e.g. [5, 2]).

Theorem 1 (Birkhoff's theorem). *The solution to vacuum spherically symmetric Einstein equations is locally isometric to the Schwarzschild solution for a certain parameter M.*

The theorem does not tell us how to fix the parameter M, that from the derivation above can be interpreted as the mass of the compact object responsible of the orbital motion. In this sense, at a given r around a center of spherical symmetry, M would correspond to the mass contained inside the sphere of radius M. In this sense, Birkhoff theorem is a relativistic counterpart of the well-known result in Newtonian's gravity of the gravitational effect of spherical distributions on point-like particles, namely:

i) The gravitational force felt by a particle inside a hollow sphere of mass M is exactly zero.

ii) The exterior effect of a spherical distribution of matter of mass M on the particle is exactly the same as the one created by a point particle of mass M at the center.

This is due, in Newtonian gravity, to the exact matching between the $\frac{1}{r^2}$ dependence of the gravitational force and the area in dependence as r^2.

This theorem can be generalized to the charged case in terms of the Reissner-Nordström solution and to solutions with cosmological constant.

5.1.4 Gravitational redshift

If we look at Schwarzschild line element, we notice that expression becomes singular at $r = 0$ and $r = 2M$, in such a way that the expression is only valid in charts with either $r \in]2M, \infty[$ or $r \in]0, 2M[$. At this stage it is not an easy question to answer if such hypersurfaces correspond to actual singularities of the geometry or to a bad choice of coordinates. Let us focus on the chart $r \in]2M, \infty[$, namely on the exterior of a spherical compact object of radius r_{matter} and connecting to infinity. As long as $r_{matter} > 2M$, the solution in the interior of the star (not vacuum) is different from Schwarzschild and problem shows up at $r = 2M$. But, if for some reason $r_{matter} < 2M$, then the chart is not covering the whole exterior. Let us focus on this situation coming from the exterior.

Specifically, at this point we are in conditions of making contact with one of the opening themes of our approach to gravitation in a relativistic setting, namely gravitational redshift. Schwarzschild solution allows us to address this issue in a systematic manner. The main two ingredients are the Schwarzschild line element (5.15) and stationarity. We proceed in two equivalent ways.

Heuristic approach

This approaches stresses the worldlines of two observers. We consider two stationary observers located at (constant) $r = r_1$ and r_2 (along the same radial direction $\theta = \theta_o$ and $\varphi = \varphi_o$ constants), with $2M < r_1 < r_2$. In Schwarschild coordinates, the spacetime trajectories of these observers are, respectively, $(t, r_1, \theta_o, \varphi_o)$ and $(t, r_2, \theta_o, \varphi_o)$, so it holds ($d\tau^2 = -ds^2$)

$$d\tau^2 = \left(1 - \frac{2M}{r}\right) dt^2 \; . \tag{5.20}$$

Evaluating for each observer, we get the relation between differentials of the proper time $d\tau_i$ as measured by each observer along its own trajectory (its "proper clock") and that of coordinate time dt_i.

$$d\tau_1 = \sqrt{1 - \frac{2M}{r_1}} dt_1 \quad , \quad d\tau_2 = \sqrt{1 - \frac{2M}{r_2}} dt_2 \ . \tag{5.21}$$

Now the observer at r_1 sends a periodic (radial) light signal to r_2 during a time lapse, that measured in its proper time is $d\tau_1$. These signals are received by observer at r_2 and (s)he measures a time lapse $d\tau_1$.

The key point in the argument is that, due to stationarity (namely no geometric feature depends on t, since ∂_t is a Killing vector) all light rays are "parallel" in the (t, r) diagram. This translates into the crucial relation

$$dt_1 = dt_2 \ , \tag{5.22}$$

so the "coordinate time" lapses, and not the "proper time" lapses, coincide. From this and (5.21) we get the relation

$$\frac{d\tau_1}{\sqrt{1 - \frac{2M}{r_1}}} = \frac{d\tau_2}{\sqrt{1 - \frac{2M}{r_2}}} \quad \Longrightarrow \quad \frac{d\tau_2}{d\tau_1} = \frac{\sqrt{1 - \frac{2M}{r_2}}}{\sqrt{1 - \frac{2M}{r_1}}} \tag{5.23}$$

Using now that the number N of "ticks" (the physical invariant quantity) emitted and received is the same, with $N = \nu_i d\tau_i$ where ν_i the frequency of the signal for each observer, we have

$$\nu_1 d\tau_1 = \nu_2 d\tau_2 \quad \Longrightarrow \quad \frac{\nu_1}{\nu_2} = \frac{d\tau_2}{d\tau_1} \ , \tag{5.24}$$

and using the expression above

$$\frac{\nu_1}{\nu_2} = \frac{\sqrt{1 - \frac{2M}{r_2}}}{\sqrt{1 - \frac{2M}{r_1}}} \ . \tag{5.25}$$

Using $r_1 < r_2$ we conclude therefore that $\nu_1 > \nu_2$, so the frequency is redshifted when light gets to larger distances.

Geometric rigorous approach

We address now the gravitational redshift discussion in a fully geometric methodology. In this approach the emphasis is on lightlike geodesics, rather than on observers trajectories, as in the previous discussion. We start by defining (e.g. [10]) the (angular) frequency ω (note the relation $\omega = 2\pi\nu$ with ν, the inverse of the period) measured by an observer u^a.

Definition 1 (Frequency measured by an observer). *Given a null geodesic affinely parametrized with tangen vector k^a (namely $k_a k^a = 0$, $\nabla_a k^a = 0$), the frequancy ω measured by an observer u^a (i.e. $u_a u^a = -1$) is given by*

$$\omega = -k^a u_a \ . \tag{5.26}$$

As a motivation for this definition, connecting a wave-like aspect (frequency) with a geometric optics one (geodescis) let us consider a wave-type equation for a massless field ϕ in Minkowski

$$\left(\frac{-\partial^2}{\partial t^2} + \Delta\right)\phi = 0 \ . \tag{5.27}$$

If we consider a mode $\phi(t,x) \sim e^{ik_a x^a} = e^{i(\omega t - \vec{k}\vec{x})}$, where $k^a = (\omega, \vec{k})$, satisfying this equation (equivalently, taking Fourier transform), we have the dispersion relation

$$k_a k^a = 0 \iff -\omega^2 + \vec{k}^2 = 0 \iff \omega = |\vec{k}| \ , \tag{5.28}$$

corresponding to a null geodesic with tangent k^a. The frequency observed by an observer stationary in this reference system $u^a = (1,0,0,0)$ is indeed $\omega = -k^a u_a$.

With these elements, let us consider two stationary observers u_1^a et u_2^a, at respective locations r_1 and r_2, $2M < r_< r_2$ collinear with one flow line of the timelike Killing vector t^a. By imposing $g_{ab}u_i^a u_i^b = -1$, we can then write along the flow line

$$u_i^a = \frac{1}{\sqrt{-t^a t_a}} t^a \bigg|_i \ , \tag{5.29}$$

At this point we consider the sending by observer u_1 of a light ray towards u_2. The light ray follows a geodesic with tangent vector k^a. Taking into account lemma 1, and t^a being a Killing vector, we know that the quantity $t^a k_a$ is constant along the geodesic. That is

$$(t^a k_a)_1 = (t^a k_a)_2 \ . \tag{5.30}$$

Writing, at each location, t^a in terms of u^a we find

$$\left(\sqrt{-t^b t_b}\ u^a k_a\right)_1 = \left(\sqrt{-t^b t_b}\ u^a k_a\right)_2 \ . \tag{5.31}$$

Using now the expression of the frequency in definition 1, we get the relation between frequencies ω_1 and ω_2

$$\left(\sqrt{-t^b t_b}\right)_1 \omega_1 = \left(\sqrt{-t^b t_b}\right)_2 \omega_2 \tag{5.32}$$

in terms of the norm of the Killing. This expression is valid in any coordinate system, in any stationary spacetime. If we consider a coordinate system in which $t^a = \partial_t$, then $t^b t_b = g_{tt}$ and

$$\left(\sqrt{-g_{tt}}\right)_1 \omega_1 = \left(\sqrt{-g_{tt}}\right)_2 \omega_2 \ , \quad \frac{\omega_1}{\omega_2} = \frac{(\sqrt{-g_{tt}})_2}{(\sqrt{-g_{tt}})_1} \tag{5.33}$$

In the particular case of Schwarzschild we finally get

$$\frac{\omega_1}{\omega_2} = \frac{\sqrt{1 - \frac{2M}{r_2}}}{\sqrt{1 - \frac{2M}{r_1}}} \ . \tag{5.34}$$

Introducing the redshift factor

$$z = \frac{\lambda_2 - \lambda_1}{\lambda_1} \ , \quad 1 + z = \frac{\lambda_2}{\lambda_1} \ , \tag{5.35}$$

we find

$$1 + z = \frac{\lambda_1}{\lambda_2} = \frac{\omega_2}{\omega_1} = \frac{\sqrt{1 - \frac{2M}{r_1}}}{\sqrt{1 - \frac{2M}{r_2}}} \ . \tag{5.36}$$

This derivation shows the remarkable geometrization effort needed to transition from the heuristic motivations in the first chapter into a sound mathematical formalism in which the phenomenon can be presented as a theorem. Indeed, a neat example illustrating the rationale underlying mathematical physics.

5.1.5 Causal structure

As we did with the Rindler spacetime, in order to gain an intuition on the causal structure of the spacetime, we look at the null geodesics, focusing on the radial ones $\theta = \theta_o$ and $\varphi = \varphi_o$. Setting to zero the line element along these trajectories we find

$$0 = f(r)dt^2 + (f(r))^{-1} dr^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 \ , \tag{5.37}$$

so that along these trajectories it is satisfied

$$\left(\frac{dt}{dr}\right)^2 = \left(1 - \frac{2M}{r}\right)^{-2} \ , \tag{5.38}$$

that is

$$\frac{dt}{dr} = \pm (f(r))^{-1} = \pm \left(1 - \frac{2M}{r}\right)^{-1} \ , \tag{5.39}$$

where the "+" signs corresponds to outgoing geodesics and the "−" to ingoing geodesics. Although we do not need an explicit solution to draw the trajectories, in this cas an explicit expression can be given, namely

$$t = \pm r_* + C \ , \tag{5.40}$$

with C a constant, and r_* the so-called *tortoise* coordinate, defined as

$$\frac{dr_*}{dr} = f^{-1} \quad , \quad r_* = r + 2M \ln\left(\frac{r}{2M} - 1\right) \tag{5.41}$$

The resulting causal structure can be see in Fig. 5.1, where outgoing and ingoing light geodesics are show (in yellow), with the corresponding light cones (in blue).

We can comment on the following features:

i) Note the different behaviour of both charts, with the interchange of "time" direction, along t for $r > 2M$ and along r forn $r < 2M$. The orientation of light cones respond to this.

ii) Outgoing light rays asymptote to slope 1 straight lines, i.e. Minkowski light rays, consistently with the recovery of a flat geometry far from the central object. This anticipates the notion of "null infinity" to be later introduced.

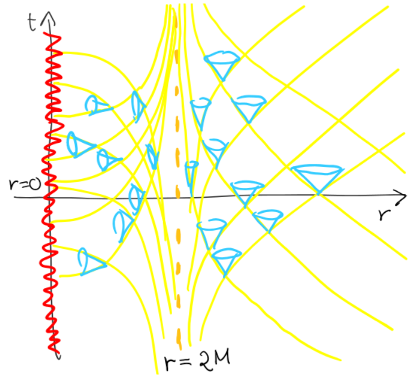

Figure 5.1: Causal structure of Schwarzschild, diagram in Schwarzschild coordinates.

iii) Light cones start for $r \to \infty$ as Minkowski light cones. As long as r decreases they close more and more and finally they degenerate at the $r = 2M$ hypersurface closing completely. Then transition in the $r < 2M$ chart to fully open cones that oriented to the left that close further and further till they fully close.

iv) Notice that the hypersurface $r = 2M$ is (asymptotically) "tangent" to the light rays, so it is "light" or null hypersurface, whereas the hypersurface $r = 0$ is encountered by "advancing" light cones (a time orientation is assumed here), that is, $r = 0$ is a spacelike hypersurface: it is "moment in time", not a "place", it happens "in future", and not "to the left" or "to the right".

Fom this picture we get an intuition of some features, but is also clear that something pathological happens in these chart representations as $r \to 2M$. We comment a bit further.

5.1.6 Some remarks about the hypersurface $r = 2M$

Null hypersurface: causal horizon

Strictly speaking, the hypersuface $r = 2M$ lays outside of the domain of charts $r > 2M$ or $r < 2M$. However it corresponds to the limit $r \to 2M$ of a family of hypersurfaces $r = c$ parametrized by the constant c, on which the vector field ∂_t is tangent. To determine the metric type of such hypersurfaces (namely the type of induced metric from the ambient spacetime metric), we must detemine the metric type of ∂_t (the other two directions, on the sphere, are always spacelike). We get

$$\boldsymbol{g}(\partial_t, \partial_t) = g_{tt} = -\left(1 - \frac{2M}{r}\right) . \tag{5.42}$$

That is, in the chart $r > 2M$, ∂_t is timelike, and hypersurfaces $r = c$ are timelike. On the contrary, for $r < 2M$ ∂_t is spacelike and hypersurfaces $r = c$ are spacelike. However, in both cases, when making the limit $\lim_{r\to 2M} \boldsymbol{g}(\partial_t, \partial_t) = 0$, indicating that $r = 2M$ is a null hypersurface.

From the discussion of the Rindler spacetime, we have learned that null hypersurfaces act as causal horizons, in the sense that if the are traversed in one sense, one cannot turn back and traverse in the other sense. The hypersurface $r = 2M$ in Schwarzschild behaves much as the hypersurface $x = 0$ behaved in Rindler. The "traversability" is however not obvious from Fig. 5.1, due to the closing of null cones: we need to "resolve" such collapse to know if a causal curve can go through $r = 2M$. This calls for looking for another representation of this geometry.

Surface of infinite redshift: geometric (apparent) horizons

Looking at the expression (5.36) for the gravitational redshift, if r_2 stays constant, the gravitational redshift grows as r_1 becomes smaller. This is consistent with the fact that light rays have "to struggle" to get away as they are emitted closer and closer to $r = 2M$: the "ticks" get distanced because the light ray is retained. Eventually, in the limit $r_1 \to 2M$, the gravitational redshift diverges

$$\lim_{r_1 \to 2M} (1+z) = \frac{\sqrt{1-\frac{2M}{r_2}}}{\sqrt{1-\frac{2M}{r_1}}} = \infty \tag{5.43}$$

Light rays cannot escape and the time "to the next tick" becomes infinite. The surface $r = 2M$ is therefore an "infinite redshift" hypersurface. This concept is related to the geometric notion of (marginally) trapped surface and apparent horizons, as we shall see later.

Time to get to the surface r=2M: "frozen stars"

In the other ingoing direction if we send a light ray from r_o to towards de horizon, we can ask how long, in coordinate time t, it takes for it get to the hypersurface $r = 2M$. According to (5.39) and choosing the sign corresponding to ingoing rays, we can write

$$dt = -\left(1 - \frac{2M}{r}\right)^{-1} dr \ . \tag{5.44}$$

Therefore

$$\begin{aligned} \Delta t &= \lim_{r\to 2M} - \int_{r_o}^{r} \left(1-\frac{2M}{r}\right)^{-1} dr = \lim_{r\to 2M} \left[r - 2M\ln(r-2M)\right]_{r_o}^{r} \\ &= (r_o - r) + 2M \ln\left(\frac{r_o - 2M}{r-2M}\right) = \infty \ . \end{aligned} \tag{5.45}$$

Therefore, it takes an infinite time for the light ray to get to the $r = 2M$. With hindsight, this is actually apparent in Fig. 5.1, since ingoing light rays have a vertical asymptote at $r = 2M$, this meaning that they cross all $t = \text{const}$ lines.

More interesting is the calculation for a radial timelike geodesic.

Exercise 2 (Frozen stars). Consider a massive test particle at rest at $r = r_o$, that stars falling in "free fall" (geodesic). Taking into account the timelike geodesic equation (5.8):

i) Show that the (Schwarzschild) coordinate time Δt that takes for the particle to $r = 2M$ is infinite.

ii) Show that the corresponding proper time $\Delta\tau$ is finite.

In other words, if an observer very far at large r's (almost Minkowskian region and therefore with a proper time that coincides with coordinate t) measures the time for the particle to reach $r = 2M$ it sees that this time is infinite. That is, from his perspective, the particle seems to slown down as it gets closer and closer to the $r = 2M$ surface. When extended to a collapsing star, it leads to the notion of a star whose surface seems to settle to $r = 2M$ and stay there: this leads to the notion of "frozen stars" in the context Openheimer-Snyder.

On the contrary, for the particle falling down, nothing of this kind happens and it gets to $r = 2M$ at a finite time. Everything seems "as normal". Whether it can get across such surface cannot be elucidated in these coordinates.

5.2 Global structure of the Schwarzschild spacetime

Points discussed about the hypersurfave $r = 2M$ strongly suggests the need to regard the surface $r = 2M$ in other coordinates. It was not an obvious step at early stages in the development general relativity to recognized the actual regular geometry nature of the $r = 2M$ hypersurface**[ref!]**.

5.2.1 Eddington-Finkelstein coordinates

Exercise 3 (Eddington-Finkelstein coordinates). Given the Schwarzschild metric in standard coordinates (t, r, θ, φ), consider the change of variables:

$$\begin{cases} t' = t + 2M \ln\left(\dfrac{r}{2M} - 1\right) \\ r' = r \\ \theta' = \theta \\ \varphi' = \varphi \end{cases}$$

i) Write the line element in the coordinates $(t', r', \theta', \varphi')$.

ii) Consider the radial outgoing and ingoing null trajectories (i.e. θ' and φ' constant). Determine the ODEs satisfied by these trajectories and sketch the corresponding outgoing and ingoing curves in a (t', r') diagram, in particular showing the null cones.

iii) Determine the coordinate time $\Delta t'$ between the emission of an ingoing radial light ray from an observer at position r_+ and its reception at r_- (with $r_+ > r_-$). If $r_- = 2M$, what can be concluded about the new coordinate system as compared with the original one?

5.2.2 Maximal extension of Schwarzschild: Kruskal coordinates

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